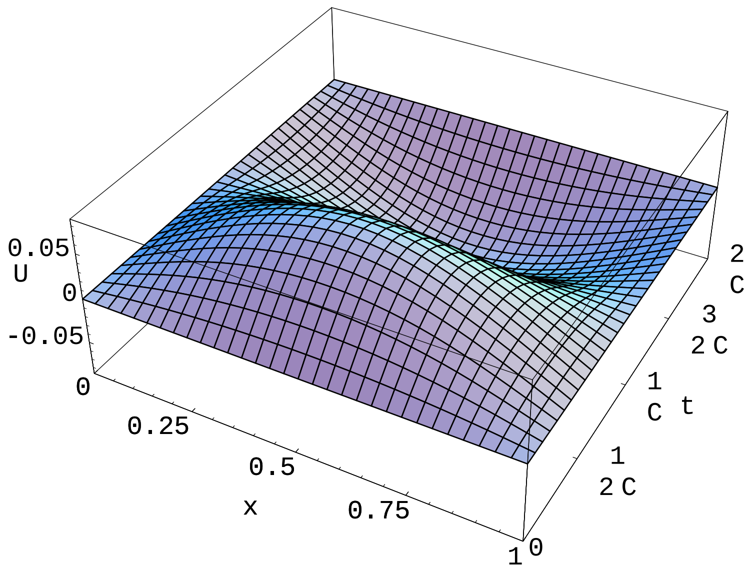
KMA354
Partial Differential Equations

Assignment 2. Due Thursday August 30, 2012

1. Using an xt diagram, solve the following problem.

[DE:]	$U_{tt} - c^2 U_{xx}$	$=$	0	$0 < x < \infty$ $0 < t < \infty$
[IC:]	$U(x,0)$	$=$	0	$0 < x < a$
			0	$a + 2l < x < \infty$
			h	$a \leq x \leq a + 2l$
	$U_t(x,0)$	$=$	0	$0 < x < \infty$
[BC:]	$U(0,t)$	$=$	0	$0 < t < \infty$

2. For the solution domain $D(0 \leq x \leq 1, t \leq \frac{2}{c})$, the vertical displacement $U(x,t)$ of a finite string is shown in the figure below. The governing equation and constraints on the string's motion are given in the following table.



[DE:]	$U_{tt} - c^2 U_{xx}$	$=$	0	$0 < x < 1$ $0 < t < \infty$
[IC:]	$U(x,0)$	$=$	0	$0 < x < 1$
	$U_t(x,0)$	$=$	$x(1-x)$	$0 < x < 1$
[BC:]	$U(0,t)$	$=$	0	$0 < t < \infty$
	$U(1,t)$	$=$	0	$0 < t < \infty$

Use an xt diagram and the method of images to find the solution $U(x,t)$ in D. Show that there are seven distinct regions within D and that the solution within each of these regions is obtained from the following left and right travelling waves defined at $t=0$. [Note that the magnitude of each travelling wave must be divided by $12c$.]

Region	$\phi(x+c\,0)$	$\psi(x-c\,0)$
$-2 \le x \le -1$		$(2+x)^2(1+2x)$
$-1 \le x \le 0$		$-x^2(3+2x)$
$0 \le x \le 1$	$x^2(3-2x)$	$-x^2(3-2x)$
$1 \le x \le 2$	$(2-x)^2(2x-1)$	
$2 \le x \le 3$	$(2-x)^2(7-2x)$	

3. Solve the nonhomogeneous wave equation

$$\frac{\partial^2 U}{\partial t^2} = 9\,\frac{\partial^2 U}{\partial x^2} - e^{-x}, \qquad 0 < x < 4,\ t > 0;$$

with boundary conditions

$$U(0,t) = U(4,t) = 0, \quad t > 0;$$

and initial conditions

$$U(x,0) = \sin(\pi x) \quad \text{and} \quad U_t(x,0) = 0, \quad 0 \le x \le 4.$$

School of Mathematics & Physics

Assignment Cover Sheet

UNIVERSITY OF TASMANIA

Student ID: ....

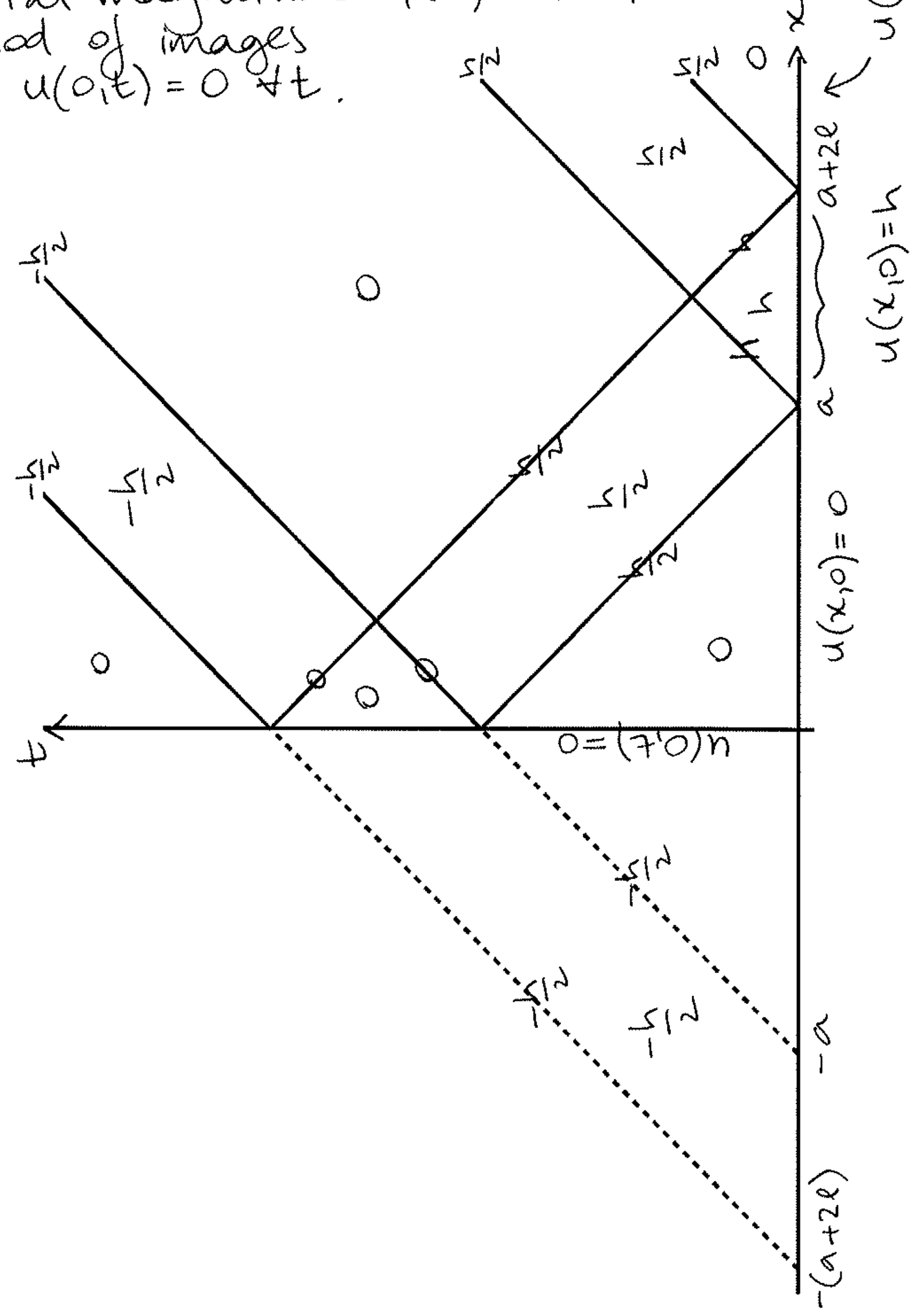
Name: ....

I declare that all material in this assignment is my own work except where there is clear acknowledgment or reference to the work of others **and** I have read the University statement on Academic Misconduct (Plagiarism) on the University website at www.utas.edu.au/plagiarism or in the Student Information Handbook.

Signed .... Date ....

1. Let $f(x) = u(x,0) = \begin{cases} h & a \le x \le a+2l \\ 0 & 0 < x < a \\ 0 & x > a+2l \end{cases}$

For $t=0$, $x>0$ every right travelling wave carries the initial magnitude $\psi(x) = f(x)/2$ and every left travelling wave carries the initial magnitude $\phi(x) = f(x)/2$.
The method of images ensures $u(0,t) = 0 \ \forall t$.



2. PDE : $u_{tt} - c^2 u_{xx} = 0 \qquad x \in (0,1), t > 0$

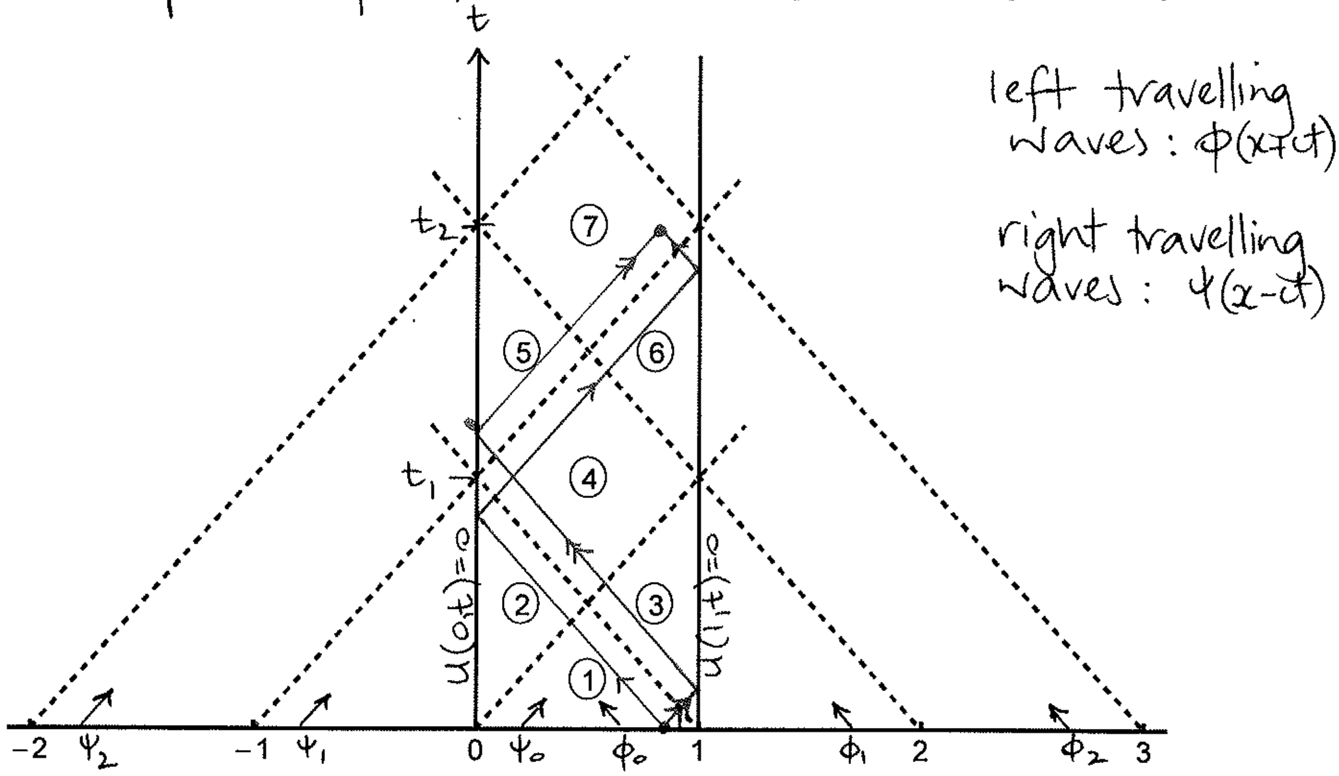
$$\left.\begin{array}{ll} \text{IC1:} & u(x,0) = 0 \\ \text{IC2:} & u_t(x,0) = x(1-x) \end{array}\right\} x \in (0,1)$$

$$\left.\begin{array}{ll} \text{BC1:} & u(0,t) = 0 \\ \text{BC2:} & u(1,t) = 0 \end{array}\right\} t > 0.$$

Since the PDE is the wave equation, the left travelling waves have speed $-c$ and the right travelling waves have speed c.

$\therefore t_1 = \frac{1}{c}$ and $t_2 = \frac{2}{c}$.

Using the method of images to enforce the boundary conditions, the left and right characteristics form regions in which the solution combines in a piecewise manner. Tracing an initial disturbance through multiple reflections at the boundaries



shows that the initial solution recurs at $t = nt_2 \quad n \in \mathbb{Z}$.

D'Alambert's solution,

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

where $u(x,0) = f(x)$
and $u_t(x,0) = g(x)$,
applies everywhere in the solution domain for an infinite string. For the finite string, this solution can only be used in a region unaffected by the boundaries; i.e. region ① for this problem.

Here, $f(x)=0$ and $g(x) = x(1-x)$.
Hence

$$u_{①}(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} s(1-s)\,ds$$

$$= \frac{1}{2c}\left[\frac{s^2}{2} - \frac{s^3}{3}\right]_{x-ct}^{x+ct}$$

$$= \frac{1}{12c}\left[s^2(3-2s)\right]_{x-ct}^{x+ct}.$$

$$= \frac{1}{12c}\left[(x+ct)^2(3-2(x+ct)) - (x-ct)^2(3-2(x-ct))\right]$$

This solution can be decomposed into respective left and right travelling waves

$$\phi_0(x+ct) = \frac{(x+ct)^2}{12c}(3-2(x+ct))$$

$$\psi_0(x-ct) = -\frac{(x-ct)^2}{12c}(3-2(x-ct))$$

which have initial definitions $(t=0)$

$$\left.\begin{aligned} \phi_0(x) &= \frac{x^2}{12c}(3-2x) \\ \psi_0(x) &= -\frac{x^2}{12c}(3-2x) \end{aligned}\right\} \quad x \in [0,1]$$

Note that

$$u(x,0) = \phi_0(x) + \psi_0(x)$$

$$= \frac{x^2}{12c}(3-2x) - \frac{x^2}{12c}(3-2x)$$

$$= 0, \quad \text{as expected.}$$

Region ②

The solution is the superposition of ϕ_0 and ψ_1 (which is yet to be defined). That is,

$$u_{②}(x,t) = \phi_0(x+ct) + \psi_1(x-ct)$$

Using the boundary condition bordering region ②,

$$\begin{aligned} u(0,t) &= 0 \\ &= \phi_0(ct) + \psi_1(-ct) \\ \Rightarrow \psi_1(-ct) &= -\phi_0(ct) \\ &= -\phi_0(-(-ct)) \end{aligned}$$

$\therefore$ In general, $\psi_1(\alpha) = -\phi_0(-\alpha)$. $\quad - \boxed{1}$

$$\begin{aligned} \Rightarrow \psi_1(x) &= -\left(\frac{(-x)^2}{12c}(3-2(-x))\right) \\ &= -\frac{x^2}{12c}(3+2x), \qquad x \in [-1,0]. \end{aligned}$$

Similarly,

$$\begin{aligned} \psi_1(x-ct) &= -\left(\frac{(-(x-ct))^2}{12c}(3-2(-(x-ct)))\right) \\ &= \frac{(x-ct)^2}{12c}(3+2(x-ct)) \end{aligned}$$

$$\therefore u_{②}(x,t) = \frac{1}{12c}\left[(x+ct)^2(3-2(x+ct)) + (x-ct)^2(3+2(x-ct))\right]$$

Region ③

The solution is the superposition of ψ_0 and ϕ_1 (which is yet to be defined). That is,

$$u_{③}(x,t) = \phi_1(x+ct) + \psi_0(x-ct).$$

Using the boundary condition bordering region ③,

$$\begin{aligned} u(1,t) &= 0 \\ &= \phi_1(1+ct) + \psi_0(1-ct) \\ \Rightarrow \phi_1(1+ct) &= -\psi_0(1-ct) \\ &= -\psi_0(2-(1+ct)) \end{aligned}$$

$\therefore$ In general, $\phi_1(\alpha) = -\psi_0(2-\alpha)$. $\quad - \boxed{2}$

$$\Rightarrow \phi_1(x) = -\left(-\frac{(2-x)^2}{12c}\left(3-2(2-x)\right)\right)$$
$$= \frac{(2-x)^2}{12c}(-1+2x) \qquad x \in [1,2]$$

$$\therefore \phi_1(x+ct) = \frac{(2-(x+ct))^2}{12c}\left(2(x+ct)-1\right)$$

and $u_{3}(x,t) = \frac{1}{12c}\left[(2-(x+ct))^2(2(x+ct)-1) - (x-ct)^2(3-2(x-ct))\right]$

Region 4

$$u_{4}(x,t) = \psi_1(x-ct) + \phi_1(x+ct)$$
$$= \frac{1}{12c}\left[(x-ct)^2(3+2(x-ct)) + (2-(x+ct))^2(2(x+ct)-1)\right]$$

Region 5

$$u_{5}(x,t) = \psi_2(x-ct) + \phi_1(x+ct)$$

Using the boundary condition bordering region 5,

$$u(0,t) = 0$$
$$= \psi_2(-ct) + \phi_1(ct)$$
$$\Rightarrow \psi_2(-ct) = -\phi_1(ct)$$
$$= -\phi_1(-(-ct))$$

$\therefore$ In general, $\psi_2(\alpha) = -\phi_1(-\alpha)$. $\quad -\boxed{3}$

$$\Rightarrow \psi_2(x) = -\left(\frac{(2-(-x))^2}{12c}(-1+2(-x))\right)$$
$$= \frac{(2+x)^2}{12c}(1+2x) \qquad x \in [-2,-1]$$

$$\therefore \psi_2(x-ct) = \frac{(2+(x-ct))^2}{12c}\left(1+2(x-ct)\right)$$

and $$u_{⑤}(x,t) = \frac{1}{12c}\Big[(2+(x-ct))^2(1+2(x-ct)) + (2-(x+ct))^2(2(x+ct)-1)\Big].$$

Region ⑥

$$u_{⑥}(x,t) = \psi_1(x-ct) + \phi_2(x+ct).$$

Using the boundary condition bordering region ⑥,

$$\begin{aligned} u(1,t) &= 0 \\ &= \psi_1(1-ct) + \phi_2(1+ct) \end{aligned}$$

$$\begin{aligned} \Rightarrow \phi_2(1+ct) &= -\psi_1(1-ct) \\ &= -\psi_1(2-(1+ct)) \end{aligned}$$

$\therefore$ In general, $\phi_2(\alpha) = -\psi_1(2-\alpha)$ — $\boxed{4}$

$$\begin{aligned} \Rightarrow \phi_2(x) &= -\left(-\frac{(2-x)^2}{12c}(3+2(2-x))\right) \\ &= \frac{(2-x)^2}{12c}(7-2x) \qquad x\in[2,3]. \end{aligned}$$

$$\therefore \phi_2(x+ct) = \frac{(2-(x+ct))^2}{12c}(7-2(x+ct))$$

and $$u_{⑥}(x,t) = \frac{1}{12c}\Big[(x-ct)^2(3+2(x-ct)) + (2-(x+ct))^2(7-2(x+ct))\Big]$$

Region ⑦

$$\begin{aligned} u_{⑦}(x,t) &= \psi_2(x-ct) + \phi_2(x+ct) \\ &= \frac{1}{12c}\Big[(2+(x-ct))^2(1+2(x-ct)) + (2-(x+ct))^2(7-2(x+ct))\Big] \end{aligned}$$

An alternative technique for determining $\psi_1(x)$, $\psi_2(x)$, $\phi_1(x)$, and $\phi_2(x)$ at $t=0$ can be seen in equations $\boxed{1}$ – $\boxed{4}$. which permit all definitions to trace back to $\psi_0(x)$ and $\phi_0(x)$.

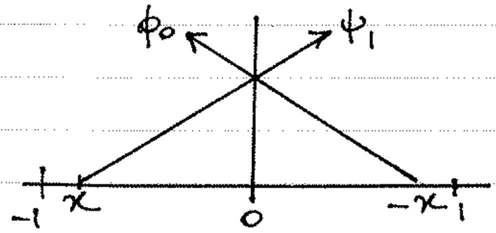
$\boxed{1}$: $\psi_1(x) = -\phi_0(-x)$ $\qquad x \in [-1, 0]$

$\boxed{2}$: $\phi_1(x) = -\psi_0(2-x)$ $\qquad x \in [1, 2]$

$\boxed{3}$: $\psi_2(x) = -\phi_1(-x)$
$\qquad = \psi_0(2+x)$ $\qquad x \in [-2, -1]$

$\boxed{4}$: $\phi_2(x) = -\psi_1(2-x)$
$\qquad = \phi_0(x-2)$ $\qquad x \in [2, 3]$.

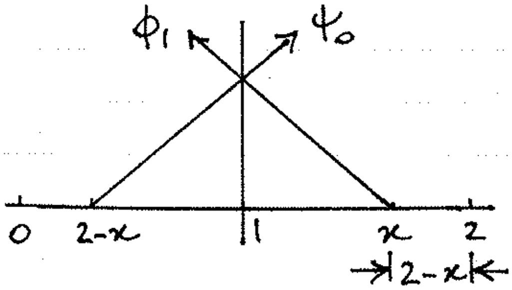
In this technique we recognise that the solution is constant along a characteristic, with magnitude defined at $t=0$. When a characteristic meets a boundary it interacts with a characteristic moving in the opposite direction, with the same speed, and having travelled the same distance.



Since the boundary condition is $u(0,t)=0$ for the case above,

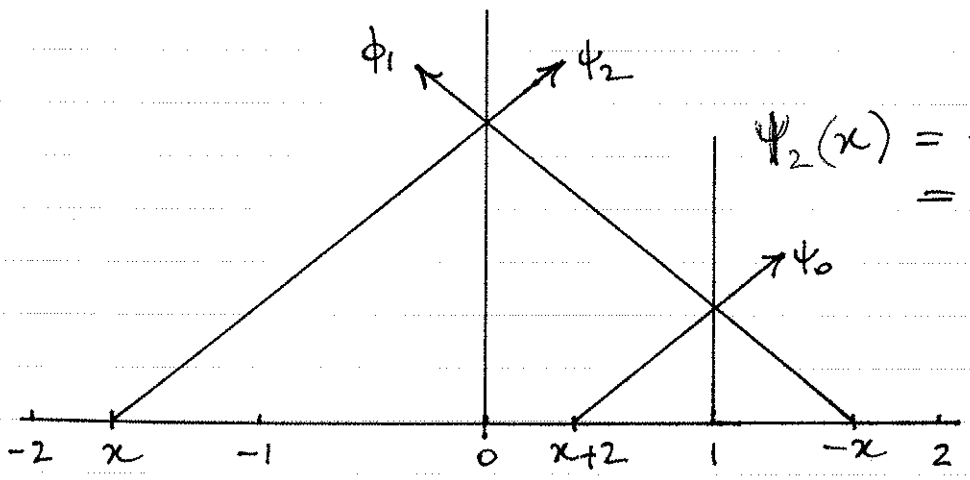
$$\psi_1(x) + \phi_0(-x) = 0$$
$$\Rightarrow \psi_1(x) = -\phi_0(-x) \qquad x \in [-1, 0]$$

For the interaction of ψ_0 and ϕ_1 at $x=1$ we must satisfy $u(1,t)=0$.



$$\phi_1(x) = -\psi_0(2-x) \qquad x \in [1,2].$$

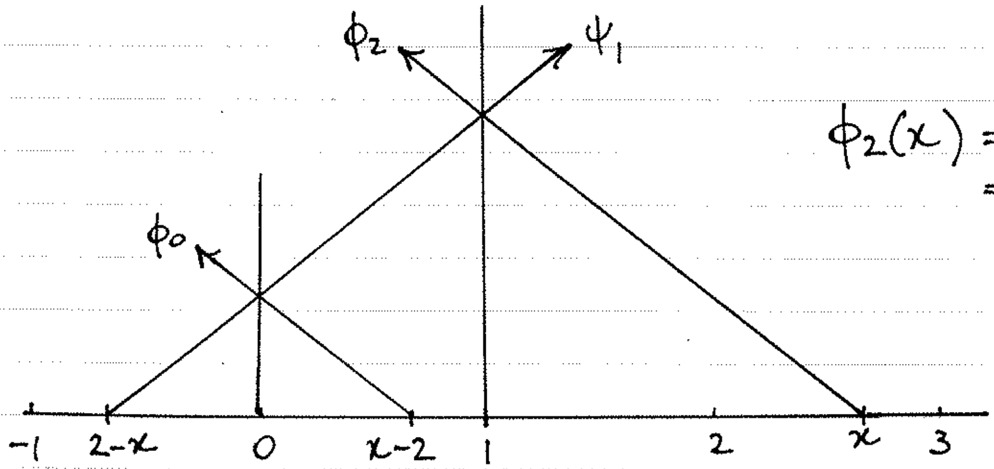
For the intersection of ψ_2 and ϕ_1 at $x=0$ we again have $u(0,t)=0$.



$$\psi_2(x) = -\phi_1(-x) = \psi_0(x+2)$$

$$x \in [-2,-1]$$

Finally, for the intersection of ψ_1 and ϕ_2 at $x=1$ we have $u(1,t)=0$.



$$\phi_2(x) = -\psi_1(2-x) = \phi_0(x-2).$$

$$x \in [2,3].$$

Definitions in further regions can be obtained similarly. For $k \geq 1$, $k \in \mathbb{Z}$

$$\begin{aligned}
\phi_{2k-1}(x) &= -\psi_0(-(x-2k)) & x &\in [2k-1, 2k] \\
\phi_{2k}(x) &= \phi_0(x-2k) & x &\in [2k, 2k+1] \\
\psi_{2k-1}(x) &= -\phi_0(-(x+2k)) & x &\in [-2k-1, -2k] \\
\psi_{2k}(x) &= \psi_0(x+2k) & x &\in [-2k, -2k+1].
\end{aligned}$$

3. PDE: $u_{tt} = 9u_{xx} - e^{-x}$ $\quad x \in (0,4), t>0.$

BC1: $u(0,t) = 0$
BC2: $u(4,t) = 0$ $\Big\}$ $t>0.$

IC1: $u(x,0) = \sin(\pi x)$
IC2: $u_t(x,0) = 0$ $\Big\}$ $x \in [0,4].$

The PDE is nonhomogeneous but it is a time independent function. As such, let

$$u(x,t) = y(x,t) + \psi(x)$$

Then

$$u_x = y_x + \psi',$$
$$u_{xx} = y_{xx} + \psi'',$$
$$u_t = y_t,$$
$$u_{tt} = y_{tt}.$$

$\therefore$ PDE: $y_{tt} = 9(y_{xx} + \psi'') - e^{-x}$

$\Rightarrow y_{tt} - 9y_{xx} = 9\psi'' - e^{-x}.$ $\quad x \in (0,4)$
$t>0.$

BC1: $y(0,t) + \psi(0) = 0$
BC2: $y(4,t) + \psi(4) = 0$ $\Big\}$ $t>0.$

IC1: $y(x,0) + \psi(x) = \sin(\pi x)$
IC2: $y_t(x,0) = 0$ $\Big\}$ $x \in [0,4].$

To split this into 2 subproblems we let

$$y_{tt} - 9y_{xx} = 0 = 9\psi'' - e^{-x},$$

$y(0,t) = 0 \Rightarrow \psi(0) = 0,$
and $y(4,t) = 0 \Rightarrow \psi(4) = 0.$

<u>Subproblem 1.</u>

ODE: $9\psi'' - e^{-x} = 0$ $\quad x \in (0,4)$
BC1: $\psi(0) = 0$
BC2: $\psi(4) = 0.$

$$\therefore \quad \psi'' = \frac{e^{-x}}{9}$$

$$\Rightarrow \quad \psi' = \int \psi'' dx = \int \frac{e^{-x}}{9} dx$$

$$= \frac{-e^{-x}}{9} + c_1$$

$$\Rightarrow \quad \psi = \int \psi' dx = \int \frac{-e^{-x}}{9} + c_1 \, dx$$

$$= \frac{e^{-x}}{9} + c_1 x + c_2$$

$c_1, c_2 \in \mathbb{R}$.

BC1: $\psi(0) = 0$

$$= \frac{e^{-0}}{9} + c_1(0) + c_2$$

$$\Rightarrow c_2 = \frac{-1}{9}$$

$$\therefore \quad \psi = \frac{1}{9}(e^{-x} - 1) + c_1 x$$

BC2: $\psi(4) = 0$

$$= \frac{1}{9}(e^{-4} - 1) + c_1 4$$

$$\Rightarrow c_1 = \frac{1 - e^{-4}}{36}$$

$$\therefore \quad \psi(x) = \frac{1}{9}\left(e^{-x} - 1 + \frac{x}{4}(1 - e^{-4})\right)$$

<u>Subproblem 2</u>

PDE: $y_{tt} - 9y_{xx} = 0 \qquad x \in (0,4), t > 0$

$$\left.\begin{array}{ll}\text{BC1:} & y(0,t) = 0 \\ \text{BC2:} & y(4,t) = 0\end{array}\right\} \quad t > 0$$

$$\left.\begin{array}{ll}\text{IC1:} & y(x,0) = \sin(\pi x) - \psi(x) \\ \text{IC2:} & y_t(x,0) = 0\end{array}\right\} \quad x \in [0,4].$$

Let $y(x,t) = X(x)T(t)$

$$\Rightarrow \quad y_{tt} = XT'' \quad \text{and} \quad y_{xx} = X''T.$$

$$\therefore \quad \frac{XT''}{9} = X''T$$

Using the homogeneous initial condition first:

$$\Rightarrow \frac{T''}{9T} = \frac{X''}{X}$$

Differentiating both sides with respect to x or t will equal zero so the expression above must equate to a constant. Based on the form of the boundary conditions, let

$$\frac{T''}{9T} = \frac{X''}{X} = -\lambda^2 < 0 \quad , \quad \lambda \in \mathbb{R}.$$

$$\therefore \quad T'' + 9\lambda^2 T = 0$$

and $\quad X'' + \lambda^2 X = 0.$

These two forms of Helmholtz's equation have the respective general solution

$$T(t) = a_1 \cos(3\lambda t) + a_2 \sin(3\lambda t)$$
$$X(x) = b_1 \cos(\lambda x) + b_2 \sin(\lambda x).$$

BC1: $\quad y(0,t) = 0$
$$= X(0)T(t)$$
$$\Rightarrow X(0) = 0 \quad \text{since } T(t) \neq 0 \ \forall t.$$
$$\therefore X(0) = b_1 \cos(0) + b_2 \sin(0) = 0$$
$$\Rightarrow b_1 = 0.$$
$$\therefore X(x) = b_2 \sin(\lambda x).$$

BC2: $\quad y(4,t) = 0$
$$= X(4)T(t)$$
$$\Rightarrow X(4) = 0 \quad \text{since } T(t) \neq 0 \ \forall t.$$
$$\therefore X(4) = b_2 \sin(\lambda 4) = 0$$
$$\Rightarrow \sin(4\lambda) = 0 \quad \text{otherwise } b_2 = 0$$
will produce the trivial solution for y.
$$\therefore \quad 4\lambda = n\pi \qquad n \in \mathbb{Z}^+$$
$$\Rightarrow \lambda = \frac{n\pi}{4}.$$

$$\therefore X_n(x) = b_n \sin\left(\frac{n\pi x}{4}\right)$$

$$T(t) = a_1 \cos(3\lambda t) + a_2 \sin(3\lambda t)$$
$$\Rightarrow T'(t) = 3\lambda(-a_1 \sin(3\lambda t) + a_2 \cos(3\lambda t)).$$

IC2: $$y_t(x,0) = 0$$
$$= T'(0)X(x)$$
$$\Rightarrow T'(0) = 0 \quad \text{since } X(x) \neq 0 \ \forall x.$$
$$\therefore T'(0) = 3\lambda(-a_1 \sin(0) + a_2 \cos(0)) = 0$$
$$\Rightarrow a_2 = 0$$
$$\therefore T(t) = a_1 \cos(3\lambda t)$$
and $$T_n(t) = a_n \cos\left(\frac{3n\pi t}{4}\right).$$

Now $$y(x,t) = \sum_{n=1}^{\infty} X_n(x) T_n(t)$$
$$= \sum_{n=1}^{\infty} a_n b_n \sin\left(\frac{n\pi x}{4}\right) \cos\left(\frac{3n\pi t}{4}\right)$$
$$= \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi x}{4}\right) \cos\left(\frac{3n\pi t}{4}\right)$$
with $\alpha_n = a_n b_n$.

IC1: $$y(x,0) = \sin(\pi x) - \psi(x)$$
$$\Rightarrow \sin(\pi x) - \psi(x) = \sum_{n=0}^{\infty} \alpha_n \sin\left(\frac{n\pi x}{4}\right)$$
$$\therefore \alpha_n = \frac{1}{2}\int_0^4 \left(\sin(\pi x) - \psi(x)\right) \sin\left(\frac{n\pi x}{4}\right) dx$$
$$= \frac{1}{2}\int_0^4 \left(\sin(\pi x) - \frac{e^{-x}}{9} + \frac{1}{9} - \frac{x}{36}(1-e^{-4})\right) \sin\left(\frac{n\pi x}{4}\right) dx$$
$$= I_1 + I_2 + I_3 + I_4$$
where
$$I_1 = \frac{1}{2}\int_0^4 \sin(\pi x) \sin\left(\frac{n\pi x}{4}\right) dx$$
$$I_2 = \frac{-1}{18}\int_0^4 e^{-x} \sin\left(\frac{n\pi x}{4}\right) dx$$
$$I_3 = \frac{1}{18}\int_0^4 \sin\left(\frac{n\pi x}{4}\right) dx$$
$$I_4 = \left(\frac{e^{-4}-1}{72}\right)\int_0^4 x \sin\left(\frac{n\pi x}{4}\right) dx$$

By the orthogonality conditions of sin,

$$I_1 = \begin{cases} 0 & n \neq 4 \\ 1 & n = 4 \end{cases}$$

Using delta function notation we can write this as

$$I_1 = \delta_4 = \begin{cases} 0 & n \neq 4 \\ 1 & n = 4 \end{cases}$$

$$I_2 = \frac{-1}{18} \int_0^4 e^{-x} \sin\left(\frac{n\pi x}{4}\right) dx$$

Using integration by parts and recurrence relationships,

$$I_2 = \frac{\frac{1}{18}\left[4e^{-x}\left(n\pi \cos\left(\frac{n\pi x}{4}\right) + 4\sin\left(\frac{n\pi x}{4}\right)\right)\right]_0^4}{16+n^2\pi^2}$$

$$= \frac{\left[4e^{-4}(n\pi\cos(n\pi) + 4\sin(n\pi)) - 4(n\pi\cos(0) + 4\sin(0))\right]}{18(16+n^2\pi^2)}$$

$$= \frac{2n\pi\left(e^{-4}\cos(n\pi) - 1\right)}{9(16+n^2\pi^2)}$$

$$= \frac{2n\pi\left(e^{-4}(-1)^n - 1\right)}{9(16+n^2\pi^2)}$$

$$I_3 = \frac{1}{18}\int_0^4 \sin\left(\frac{n\pi x}{4}\right) dx$$

$$= \frac{1}{18}\left(-\frac{4}{n\pi}\right)\left[\cos\left(\frac{n\pi x}{4}\right)\right]_0^4$$

$$= \frac{-2}{9n\pi}\left[\cos(n\pi) - \cos(0)\right]$$

$$= \frac{-2}{9n\pi}\left[(-1)^n - 1\right]$$

$$= \begin{cases} 0 & n \text{ even} \\ \frac{4}{9n\pi} & n \text{ odd} \end{cases}$$

$$I_4 = \left(\frac{e^{-4}-1}{72}\right)\int_0^4 x \sin\left(\frac{n\pi x}{4}\right)dx$$

Let $u = x$ and $dv = \sin\left(\frac{n\pi x}{4}\right)dx$

$\Rightarrow$ $du = dx$ and $v = \frac{-4}{n\pi}\cos\left(\frac{n\pi x}{4}\right)$

$$\therefore I_4 = \left(\frac{e^{-4}-1}{72}\right)\left[\frac{-4x}{n\pi}\cos\left(\frac{n\pi x}{4}\right) + \frac{4}{n\pi}\int \cos\left(\frac{n\pi x}{4}\right)dx\right]_0^4$$

$$= \left(\frac{e^{-4}-1}{72}\right)\left[-\frac{4x}{n\pi}\cos\left(\frac{n\pi x}{4}\right) + \left(\frac{4}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{4}\right)\right]_0^4$$

$$= \left(\frac{e^{-4}-1}{72}\right)\left[\left(\frac{-16}{n\pi}\cos(n\pi) + \frac{16}{n^2\pi^2}\sin(n\pi)\right) - \left(0 + \left(\frac{4}{n\pi}\right)^2 \sin(0)\right)\right]$$

$$= \frac{2(1-e^{-4})}{9n\pi}\cos(n\pi)$$

$$= \frac{2(1-e^{-4})}{9n\pi}(-1)^n$$

Combining $I_1 \to I_4$ for all n,

$$\alpha_n = \delta_4 + \frac{2n\pi\left(e^{-4}(-1)^n-1\right)}{9(16+n^2\pi^2)} - \frac{2}{9n\pi}\left[(-1)^n-1\right] + \frac{2(1-e^{-4})(-1)^n}{9n\pi}$$

$$= \delta_4 + \frac{2}{9}\left[\frac{n\pi\left(e^{-4}(-1)^n-1\right)}{(16+n^2\pi^2)} + \frac{1-(-1)^n+(-1)^n(1-e^{-4})}{n\pi}\right]$$

$$= \delta_4 + \frac{2}{9}\left[\frac{n\pi\left(e^{-4}(-1)^n-1\right)}{16+n^2\pi^2} + \frac{1-e^{-4}(-1)^n}{n\pi}\right]$$

$$= \delta_4 + \frac{2}{9}\left[\frac{n^2\pi^2\left(e^{-4}(-1)^n-1\right) - (16+n^2\pi^2)\left(e^{-4}(-1)^n-1\right)}{n\pi(16+n^2\pi^2)}\right]$$

$$= \delta_4 + \frac{2}{9}\left(e^{-4}(-1)^n-1\right)\left(\frac{n^2\pi^2-(16+n^2\pi^2)}{n\pi(16+n^2\pi^2)}\right)$$

$$= \delta_4 + \frac{32\left(1-e^{-4}(-1)^n\right)}{9n\pi(16+n^2\pi^2)}$$

$$\therefore \quad y(x,t) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi x}{4}\right)\cos\left(\frac{3n\pi t}{4}\right)$$

$$= \sum_{n=1}^{\infty}\left(\delta_4 + \frac{32\left(1-e^{-4}(-1)^n\right)}{9n\pi\left(16+n^2\pi^2\right)}\right)\sin\left(\frac{n\pi x}{4}\right)\cos\left(\frac{3n\pi t}{4}\right)$$

$$= \sin(\pi x)\cos(3\pi t)$$

$$+ \frac{32}{9\pi}\sum_{n=1}^{\infty}\left(\frac{1-e^{-4}(-1)^n}{(16+n^2\pi^2)n}\right)\sin\left(\frac{n\pi x}{4}\right)\cos\left(\frac{3n\pi t}{4}\right)$$

Finally, $\quad u(x,t) = y(x,t) + \psi(x)$

$$= \frac{1}{9}\left(e^{-x} - 1 + \frac{x}{4}\left(1-e^{-4}\right)\right) + \sin(\pi x)\cos(3\pi t)$$

$$+ \frac{32}{9\pi}\sum_{n=1}^{\infty}\left(\frac{1-e^{-4}(-1)^n}{(16+n^2\pi^2)n}\right)\sin\left(\frac{n\pi x}{4}\right)\cos\left(\frac{3n\pi t}{4}\right)$$