KMA354
Partial Differential Equations

Assignment 4. Due any time between now and the last day of semester 2.

1. Hermite's differential equation is $U'' - 2x\,U' + 2n\,U = 0$.

The Hermite polynomials, $H_n(x)$, are solutions to the Hermite equation and may be defined by the generating function

$$G(t,x) \;=\; e^{2tx-t^2} \;=\; \sum_{n=0}^{\infty} H_n(x)\,\frac{t^n}{n!}\,.$$

Obtain relationships between Hermite polynomials by

(i) differentiating the generating function with respect to t;

(ii) differentiating the generating function with respect to x.

2. Consider the Sturm-Liouville boundary value problem

$$U'' + \lambda\,U = 0\ , \qquad U(0) = 0\ , \qquad \text{and} \qquad U(1) = 0\ .$$

(i) Determine the eigenvalues, eigenfunctions, and normalised eigenfunctions.

(ii) Repeat (i) but with the second boundary condition changed to $U'(1) = 0$.

3. Solve the following nonhomogeneous boundary value problems using the Green's function technique.

(i) $U'' = -x$
$U(0) = 0\ ,\ U(1) = 0$.

(ii) $U'' + U = -1$
$U(0) = 0\ ,\ U(\pi/2) = 0$.

(iii) $U'' = -2\,x$
$U(0) = 0\ ,\ U(1) + U'(1) = 2$.

(iv) $U'' = -2\,x$
$U(0) = 2\ ,\ U(1) + U'(1) = 0$.
You may reference work from (iii).

4. Use a Fourier series method to solve the following two dimensional boundary value problem.

$$\begin{array}{ll}
\nabla^2 u = 0 & 0 < \phi \le 2\pi\ ,\ b < r < c\ , \\
u(b,\phi) = f(\phi) & 0 < \phi \le 2\pi\ , \\
u(c,\phi) = 0 & 0 < \phi \le 2\pi\ .
\end{array}$$

$$u'' - 2xu' + 2nu = 0$$

$$G(t,x) = \exp(2tx - t^2) = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$$

(i) $$\frac{\partial}{\partial t} G(t,x) = (2x - 2t) \exp(2tx - t^2)$$

$$= (2x - 2t)\, G(t,x). \quad \text{— (1)}$$

$$\frac{\partial}{\partial t} G(t,x) = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} \frac{\partial t^n}{\partial t}$$

$$= \sum_{n=1}^{\infty} \frac{H_n(x)}{n!}\, n t^{n-1}$$

$$= \sum_{n=1}^{\infty} H_n(x) \frac{t^{n-1}}{(n-1)!}$$

$$= \sum_{n=0}^{\infty} H_{n+1}(x) \frac{t^n}{n!} \quad \text{— (2)}$$

Equating (1) and (2),

$$(2x - 2t)\, G(t,x) = \sum_{n=0}^{\infty} H_{n+1}(x) \frac{t^n}{n!}$$

$$\Rightarrow (2x - 2t) \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} H_{n+1}(x) \frac{t^n}{n!}$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[(2x - 2t) H_n(x) \frac{t^n}{n!} - H_{n+1}(x) \frac{t^n}{n!} \right] = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[\left(2x H_n(x) - H_{n+1}(x)\right) \frac{t^n}{n!} - 2 H_n(x) \frac{t^{n+1}}{n!} \right] = 0$$

$$\Rightarrow \left(2x H_0(x) - H_1(x)\right) \frac{t^0}{0!}$$

$$+ \sum_{n=1}^{\infty} \left(\frac{2x H_n(x) - H_{n+1}(x)}{n!} - \frac{2 H_{n-1}(x)}{(n-1)!} \right) t^n = 0$$

By inspection we see that

$$H_1(x) = 2x H_0(x)$$

and since $t^n \neq 0 \quad \forall t, n$

$$\frac{2x H_n(x) - H_{n+1}(x)}{n!} - \frac{2H_{n-1}(x)}{(n-1)!} = 0 \quad , n \geq 1 .$$

$$\Rightarrow \quad 2x H_n(x) - H_{n+1}(x) - 2n H_{n-1}(x) = 0$$

$$\Rightarrow \quad H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x) \ , \ n \geq 1 .$$

(ii) $$\frac{\partial}{\partial x} G(t,x) = 2t \exp(2tx - t^2)$$

$$= 2t \, G(t,x)$$

$$= 2t \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \qquad - (3)$$

$$\frac{\partial}{\partial x} G(t,x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\partial}{\partial x} H_n(x)$$

$$= \sum_{n=0}^{\infty} H_n'(x) \frac{t^n}{n!} \qquad - (4)$$

Equating (3) and (4),

$$\sum_{n=0}^{\infty} 2 H_n(x) \frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty} H_n'(x) \frac{t^n}{n!}$$

$$\Rightarrow \sum_{n=0}^{\infty} 2H_n(x) \frac{t^{n+1}}{n!} - H_n'(x) \frac{t^n}{n!} = 0$$

$$\Rightarrow -H_0'(x) \frac{t^0}{0!} + \sum_{n=1}^{\infty} \left(\frac{2H_{n-1}(x)}{(n-1)!} - \frac{H_n'(x)}{n!} \right) t^n = 0$$

Since $t^n \neq 0 \quad \forall\, n, t$

$$H_0'(x) = 0$$

and $\dfrac{2H_{n-1}(x)}{(n-1)!} = \dfrac{H_n'(x)}{n!}$, $n \geq 1$.

$$\Rightarrow H_n'(x) = 2n\, H_{n-1}(x) \quad , n \geq 1.$$

2. $u'' + \lambda u = 0 \qquad u(0) = 0, \quad u(1) = 0.$

The equation in self adjoint form is

$$\frac{d}{dx}\left[1 \frac{du}{dx}\right] + \lambda u = 0$$

with $H(x) = 1$, $Q(x) = 0$, $R(x) = 1$,

and eigenvalues λ.

The general solution is

$$u(x) = a_1 \cos(\sqrt{\lambda}\,x) + a_2 \sin(\sqrt{\lambda}\,x)$$

(i) using boundary condition 1:

$$u(0) = 0$$

$$\Rightarrow \quad a_1 \cos(0) + a_2 \sin(0) = 0$$

$$\Rightarrow \quad a_1 = 0.$$

$$\therefore \quad u = a_2 \sin(\sqrt{\lambda}\,x).$$

using boundary condition 2:

$$u(1) = 0$$

$$\Rightarrow \quad a_2 \sin(\sqrt{\lambda}) = 0$$

$$\Rightarrow \quad \sqrt{\lambda} = n\pi \qquad n = 1, 2, 3, \ldots$$

$\therefore \; u_n(x) = a_n \sin(n\pi x)$ eigenfunctions

and $\lambda_n = n^2\pi^2$ eigenvalues.

The normalised eigenfunction is

$$\phi_n = \frac{u_n}{\| u_n \|_2}$$

where $\| u_n \|_2 = \sqrt{\int_a^b u_n^2(x) R(x)\, dx}$

$$= \sqrt{\int_0^1 a_n^2 \sin^2(n\pi x)\, dx}$$

$$= a_n \sqrt{\int_0^1 \frac{1 - \cos(2n\pi x)}{2}\, dx}$$

$$= \frac{a_n}{\sqrt{2}} \sqrt{\left[x - \frac{\sin(2n\pi x)}{2n\pi} \right]_0^1}$$

$$= \frac{a_n}{\sqrt{2}} \sqrt{(1-0) - \frac{(\sin(2n\pi) - 0)}{2n\pi}}$$

$$= \frac{a_n}{\sqrt{2}}$$

$$\therefore \phi_n = \frac{a_n \sin(n\pi x)}{a_n / \sqrt{2}}$$

$$= \sqrt{2} \sin(n\pi x) \qquad n = 1, 2, 3, \ldots$$

(ii) boundary condition 1 is unchanged so we still have

$$u = a_2 \sin(\sqrt{\lambda}\, x).$$

using boundary condition 2:

$$u'(1) = 0$$

$$\Rightarrow \sqrt{\lambda}\, a_2 \cos(\sqrt{\lambda}\, x)\Big|_{x=1} = 0$$

$$\Rightarrow \sqrt{\lambda}\, a_2 \cos(\sqrt{\lambda}) = 0.$$

Since $\lambda, a_2 \neq 0$ we must have

$\cos(\sqrt{\lambda}) = 0$

$$\Rightarrow \sqrt{\lambda} = (2n+1)\frac{\pi}{2} \qquad n = 0, 1, 2, \ldots$$

$$\therefore u_n(x) = a_n \sin\left((2n+1)\frac{\pi x}{2}\right) \qquad n = 0, 1, 2 \ldots$$

are the eigenfunctions, and

$$\lambda_n = \left((2n+1)\frac{\pi}{2}\right)^2 \quad \text{are the}$$

eigenvalues.

The normalised eigenfunction is

$$\phi_n = \frac{u_n}{\|u_n\|_2}$$

where $\|u_n\|_2 = \sqrt{\int_a^b u_n^2(x) R(x)\, dx}$

$$= \sqrt{\int_0^1 a_n^2 \sin^2\left((2n+1)\frac{\pi x}{2}\right) dx}$$

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$$= \frac{a_n}{\sqrt{2}} \sqrt{\int_0^1 1 - \cos\left(\left(2n+1\right)\pi x\right) dx}$$

$$= \frac{a_n}{\sqrt{2}} \sqrt{\left[x - \frac{\sin\left(\left(2n+1\right)\pi x\right)}{(2n+1)\pi} \right]_0^1}$$

$$= \frac{a_n}{\sqrt{2}} \sqrt{(1-0) - \frac{\sin\left(\left(2n+1\right)\pi\right) - 0}{(2n+1)\pi}}$$

$$= \frac{a_n}{\sqrt{2}}$$

$$\therefore \phi_n = \frac{a_n \sin\left(\left(2n+1\right)\frac{\pi x}{2}\right)}{a_n / \sqrt{2}}$$

$$= \sqrt{2} \sin\left(\left(2n+1\right)\frac{\pi x}{2}\right) \qquad n = 0, 1, 2, \ldots$$

3 Two point boundary value problems

$$\frac{d}{dx}\left[H(x)\frac{du}{dx}\right] + Q(x)u = -f(x)$$

$$-\mu_1 u'(a) + \sigma_1 u(a) = \alpha$$
$$\mu_2 u'(b) + \sigma_2 u(b) = \beta.$$

The Green function $I(x,\xi)$ is the solution to

1. $$\frac{d}{dx}\left[H(x)I\right] + Q(x)I = 0 \qquad x \neq \xi.$$

2. $$\left.\begin{aligned} -\mu_1 \frac{dI}{dx}\Big|_{x=a} + \sigma_1 I\Big|_{x=a} &= 0. \\ \mu_2 \frac{dI}{dx}\Big|_{x=b} + \sigma_2 I\Big|_{x=b} &= 0. \end{aligned}\right\} \quad x \neq \xi.$$

3. $$I\Big|_{x\to\xi+} - I\Big|_{x\to\xi-} = 0. \qquad x = \xi.$$

4. $$\frac{dI}{dx}\Big|_{x\to\xi+} - \frac{dI}{dx}\Big|_{x\to\xi-} = -\frac{1}{H(\xi)} \qquad x = \xi.$$

Then $I(x,\xi) = I(\xi,x)$.

The solution to the BVP is

$$u(x) = \int_a^x I(x,\xi)f(\xi)\,d\xi + \int_x^b I(x,\xi)f(\xi)\,d\xi + \frac{H(a)}{\mu_1}\alpha I(x,a) + \frac{H(b)}{\mu_2}\beta I(x,b)$$

If $\mu_1 = 0$, replace $\frac{I(x,a)}{\mu_1}$ with $\frac{1}{\sigma_1}\frac{\partial I}{\partial \xi}(x,a)$

If $\mu_2 = 0$, replace $\frac{I(x,b)}{\mu_2}$ with $-\frac{1}{\sigma_2}\frac{dI}{d\xi}(x,b)$.

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(i) $u'' = -x$

$u(0) = 0, \quad u(1) = 0.$

In self-adjoint form the PDE is written

$$\frac{d}{dx}\left[1 \frac{du}{dx}\right] + 0u = -x$$

and by inspection $H(x)=1$, $Q(x)=0$, and $f(x) \equiv x$.

In the required mixed form, the boundary conditions are

$$0\,u'(0) + 1\,u(0) = 0$$
$$0\,u'(1) + 1\,u(1) = 0$$

$\therefore \mu_1 = 0, \quad \sigma_1 = 1, \quad a = 0, \quad \alpha = 0.$
$\mu_2 = 0, \quad \sigma_2 = 1, \quad b = 1, \quad \beta = 0.$

Constructing the Green function:

1. $$\frac{d}{dx}\left[1 \frac{dI}{dx}\right] + 0u = 0$$

$$\Rightarrow \quad I'' = 0$$

$$\Rightarrow \quad I(x,\xi) = \begin{cases} A(\xi)x + B(\xi) & x<\xi \\ C(\xi)x + D(\xi) & x>\xi. \end{cases}$$

2. $I(0,\xi) = 0$

$\Rightarrow \quad A(\xi)\,0 + B(\xi) = 0$

$\Rightarrow \quad B(\xi) = 0.$

$I(1,\xi) = 0$

$\Rightarrow \quad C(\xi)\,1 + D(\xi) = 0$

$\Rightarrow \quad D(\xi) = -C(\xi).$

$$\therefore I(x,\xi) = \begin{cases} A(\xi)x & x<\xi \\ C(\xi)(x-1) & x>\xi. \end{cases}$$

3. $$I\big|_{x\to\xi+} - I\big|_{x\to\xi-} = 0$$

$$\Rightarrow C(\xi)(\xi^{+}-1) - A(\xi)\xi^{-} = 0$$

$$\Rightarrow C(\xi) = \frac{A(\xi)\xi}{(\xi-1)}$$

$$\therefore I(x,\xi) = \begin{cases} A(\xi)x & x \le \xi. \\ \frac{A(\xi)\xi}{(\xi-1)}(x-1) & x \ge \xi. \end{cases}$$

4. $$\left.\frac{dI}{dx}\right|_{x\to\xi+} - \left.\frac{dI}{dx}\right|_{x\to\xi-} = \frac{-1}{H(\xi)}$$

$$\Rightarrow \left.\frac{A(\xi)\xi}{\xi-1}\right|_{x\to\xi+} - \left.A(\xi)\right|_{x\to\xi-} = -\frac{1}{1}$$

$$\Rightarrow A(\xi)\left(\frac{\xi}{\xi-1} - 1\right) = -1$$

$$\Rightarrow A(\xi)\left(\frac{1}{\xi-1}\right) = -1$$

$$\Rightarrow A(\xi) = 1-\xi.$$

$$\therefore I(x,\xi) = \begin{cases} (1-\xi)x & x \le \xi \\ (1-x)\xi & x \ge \xi. \end{cases}$$

Since $\alpha = \beta = 0$,

$$u(x) = \int_0^x I(x,\xi)f(\xi)d\xi + \int_x^1 I(x,\xi)f(\xi)d\xi.$$

$$= \int_0^x (1-x)\xi\,\xi\,d\xi + \int_x^1 (1-\xi)x\,\xi\,d\xi.$$

$$= (1-x)\int_0^x \xi^2 d\xi + x\int_x^1 (\xi-\xi^2)d\xi$$

$$= (1-x)\frac{x^3}{3} + x\left[\left(\frac{1-x^2}{2}\right) - \left(\frac{1-x^3}{3}\right)\right]$$

$$= \frac{x^3}{3} - \frac{x^4}{3} + \frac{x}{2} - \frac{x^3}{2} - \frac{x}{3} + \frac{x^4}{3}$$
$$= \frac{x}{6} - \frac{x^3}{6}$$
$$= \frac{x}{6}(1 - x^2).$$

(ii) $u'' + u = -1$
$u(0) = 0, \quad u(\pi/2) = 0.$

In the required form the BVP is written

$$\frac{d}{dx}\left[1 \frac{du}{dx}\right] + 1\,u = -1$$
$$0\,u'(0) + 1\,u(0) = 0$$
$$0\,u'(\pi/2) + 1\,u(\pi/2) = 0.$$

By inspection, $H(x) = 1$, $Q(x) = 1$, $f(x) = 1$.
$\mu_1 = 0$, $\sigma_1 = 1$, $a = 0$, $\alpha = 0$
$\mu_2 = 0$, $\sigma_2 = 1$, $b = \pi/2$, $\beta = 0$.

Constructing the Green function:

1. $I'' + I = 0$

$$\Rightarrow I(x,\xi) = \begin{cases} A(\xi)\cos(x) + B(\xi)\sin(x) & x<\xi \\ C(\xi)\cos(x) + D(\xi)\sin(x) & x>\xi \end{cases}$$

2. $I(0) = 0$
$\Rightarrow A(\xi)\cos(0) + B(\xi)\sin(0) = 0$
$\Rightarrow A(\xi) = 0.$
$I(\pi/2) = 0$
$\Rightarrow C(\xi)\cos(\pi/2) + D(\xi)\sin(\pi/2) = 0$
$\Rightarrow D(\xi) = 0.$

$$\therefore \quad I(x,\xi) = \begin{cases} B(\xi)\sin(x) & x<\xi \\ C(\xi)\cos(x) & x>\xi \end{cases}$$

3. $$I\big|_{x\to\xi^+} - I\big|_{x\to\xi^-} = 0$$

$$\Rightarrow C(\xi)\cos(\xi^+) - B(\xi)\sin(\xi^-) = 0$$
$$\Rightarrow C(\xi) = B(\xi)\tan(\xi).$$

$$I(x,\xi) = \begin{cases} B(\xi)\sin(x) & x \le \xi \\ B(\xi)\tan(\xi)\cos(x) & x \ge \xi. \end{cases}$$

4. $$\frac{dI}{dx}\bigg|_{x\to\xi^+} - \frac{dI}{dx}\bigg|_{x\to\xi^-} = -\frac{1}{H(\xi)}$$

$$\Rightarrow -B(\xi)\tan(\xi)\sin(\xi^+) - B(\xi)\cos(\xi^-) = -1$$

$$\Rightarrow B(\xi)\left(\frac{\sin^2(\xi)}{\cos(\xi)} + \cos(\xi)\right) = 1.$$

$$\Rightarrow B(\xi)\left(\frac{\sin^2(\xi) + \cos^2(\xi)}{\cos(\xi)}\right) = 1$$

$$\Rightarrow B(\xi) = \cos(\xi).$$

$$I(x,\xi) = \begin{cases} \cos(\xi)\sin(x) & x \le \xi \\ \sin(\xi)\cos(x) & x \ge \xi. \end{cases}$$

Since $\alpha = \beta = 0$,

$$u(x) = \int_0^x I(x,\xi) f(\xi)\,d\xi + \int_x^{\pi/2} I(x,\xi) f(\xi)\,d\xi.$$

$$= \int_0^x \sin(\xi)\cos(x)\,1\,d\xi + \int_x^{\pi/2}\cos(\xi)\sin(x)\,1\,d\xi.$$

$$= \cos(x)\int_0^x \sin(\xi)\,d\xi + \sin(x)\int_x^{\pi/2}\cos(\xi)\,d\xi.$$

$$= -\cos(x)\Big[\cos(\xi)\Big]_0^x + \sin(x)\Big[\sin(\xi)\Big]_x^{\pi/2}$$

$$= -\cos(x)\left[\cos(x) - 1\right] + \sin(x)\left[\sin\left(\tfrac{\pi}{2}\right) - \sin(x)\right]$$

$$= -\cos^2(x) - \sin^2(x) + \cos(x) + \sin(x)$$

$$= \cos(x) + \sin(x) - 1.$$

(iii) $u'' = -2x$

$u(0) = 0, \quad u(1) + u'(1) = 2$

In the required format the BVP is written

$$\frac{d}{dx}\left[1 \frac{du}{dx}\right] + 0\,u = -2x$$

$$0\, u'(0) + 1\, u(0) = 0$$

$$1\, u'(1) + 1\, u(1) = 2.$$

By inspection, $H(x) = 1$, $Q(x) = 0$, $f(x) = 2x$,

$\mu_1 = 0, \quad \sigma_1 = 1, \quad a = 0, \quad \alpha = 0.$

$\mu_2 = 1, \quad \sigma_2 = 1, \quad b = 1, \quad \beta = 2.$

Constructing the Green function:

1. $I'' = 0$

$$I(x,\xi) = \begin{cases} A(\xi)x + B(\xi) & x < \xi. \\ C(\xi)x + D(\xi) & x > \xi \end{cases}$$

2. $I(0) = 0$

$\Rightarrow \quad A(\xi)\,0 + B(\xi) = 0$

$\Rightarrow \quad B(\xi) = 0.$

$$\left.\frac{dI}{dx}\right|_{x=1} + I(1) = 0$$

$$\Rightarrow \left(C(\xi)\Big|_{x=1}\right) + \left(C(\xi)\,1 + D(\xi)\right) = 0$$

$\Rightarrow \quad 2\,C(\xi) + D(\xi) = 0$

$\Rightarrow \quad D(\xi) = -2\,C(\xi).$

$$I(x,\xi) = \begin{cases} A(\xi)x & x < \xi \\ C(\xi)(x-2) & x > \xi \end{cases}$$

3. $I|_{x\to\xi^+} - I|_{x\to\xi^-} = 0$

$\Rightarrow \quad C(\xi)(\xi^+ - 2) - A(\xi)\,\xi^- = 0$

$$\Rightarrow \quad c(\xi) = A(\xi)\frac{\xi}{\xi-2}.$$

$$\therefore I(x,\xi) = \begin{cases} A(\xi)\,x & x \le \xi \\ A(\xi)\,\dfrac{(x-2)\xi}{(\xi-2)} & x \ge \xi. \end{cases}$$

4. $\left.\dfrac{dI}{dx}\right|_{x\to\xi^+} - \left.\dfrac{dI}{dx}\right|_{x\to\xi^-} = -\dfrac{1}{H(\xi)}$

$$\Rightarrow A(\xi)\left.\left(\frac{\xi}{\xi-2}\right)\right|_{x\to\xi^+} - \left.A(\xi)\right|_{x\to\xi^-} = -1$$

$$\Rightarrow A(\xi)\left(\frac{\xi}{\xi-2} - 1\right) = -1$$

$$\Rightarrow A(\xi)\left(\frac{2}{\xi-2}\right) = -1$$

$$\Rightarrow A(\xi) = \frac{2-\xi}{2}.$$

$$\therefore I(x,\xi) = \begin{cases} (2-\xi)\dfrac{x}{2} & x \le \xi \\ (2-x)\dfrac{\xi}{2} & x \ge \xi. \end{cases}$$

Since $\alpha = 0$ and $\beta \neq 0$,

$$u(x) = \int_0^x I(x,\xi)f(\xi)\,d\xi + \int_x^1 I(x,\xi)f(\xi)\,d\xi + \frac{H(1)}{\mu_2}\beta\, I(x,1)$$

$$= \int_0^x (2-x)\frac{\xi}{2}\,2\xi\,d\xi + \int_x^1 (2-\xi)\frac{x}{2}\,2\xi\,d\xi + 2\left.(2-\xi)\frac{x}{2}\right|_{\xi=1}$$

$$= (2-x)\int_0^x \xi^2\,d\xi + x\int_x^1 (2\xi-\xi^2)\,d\xi + x$$

$$= (2-x)\frac{x^3}{3} + x\left[\xi^2 - \frac{\xi^3}{3}\right]_x^1 + x$$

$$= \frac{2x^3}{3} - \frac{x^4}{3} + x(1-x^2) - x\left(\frac{1}{3} - \frac{x^3}{3}\right) + x$$

$$= \left\{\frac{2x^3}{3} - \cancel{\frac{x^4}{3}} + x - x^3 - \frac{x}{3} + \cancel{\frac{x^4}{3}}\right\} + x \quad — (*)$$

$$= -\frac{x^3}{3} + \frac{5x}{3}$$

$$= \frac{x}{3}(5 - x^2)$$

(iv) $u'' = -2x$

$u(0) = 2, \quad u(1) + u'(1) = 0$

The only difference between (iii) and (iv) is that the values of α and β have swapped. So the Green function will be the same and the integral component of the solution will be the same, but $\frac{H(b)}{\mu_2}\beta I(x,b)$ must be replaced by $\frac{H(a)}{\sigma_1}\alpha \frac{\partial I}{\partial \xi}(x,a)$ (since $\mu_1 = 0$).

In equation (*) above, the section in the braces is the solution to the integrals.

$$\therefore u(x) = \left\{-\frac{x^3}{3} + \frac{2x}{3}\right\} + \frac{1}{1} \times 2 \times \left(\frac{2-x}{2}\right)\Big|_{\xi=a=0}$$

$$= -\frac{x^3}{3} + \frac{2x}{3} + 2 - x$$

$$= 2 - \frac{x}{3} - \frac{x^3}{3}$$

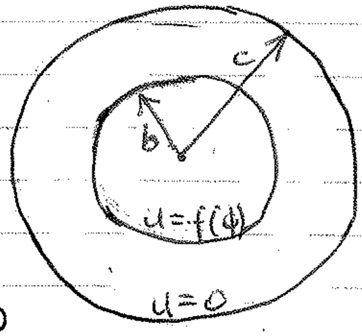
$$= \frac{1}{3}(6 - x - x^3).$$

4 $\nabla^2 u = 0 \qquad 0 < \phi \le 2\pi \qquad b < r < c$

$u(b,\phi) = f(\phi) \qquad 0 < \phi \le 2\pi$

$u(c,\phi) = 0 \qquad 0 < \phi \le 2\pi .$

$$u \equiv u(r,\phi)$$



In cylindrical coordinates,

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} = 0$$

Let $u = R(r)\Phi(\phi)$

$$\Rightarrow \nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial (R\Phi)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}(R\Phi) = 0$$

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\Phi\frac{dR}{dr}\right) + \frac{1}{r^2}R\frac{d^2\Phi}{d\phi^2} = 0$$

$$\Rightarrow \frac{\Phi}{r}\left(r\frac{d^2R}{dr^2} + \frac{dR}{dr}\right) + \frac{R}{r^2}\frac{d^2\Phi}{d\phi^2} = 0 .$$

dividing through by $R\Phi$

$$\Rightarrow \frac{1}{rR}\left(r\frac{d^2R}{dr^2} + \frac{dR}{dr}\right) + \frac{1}{r^2}\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0$$

$$\Rightarrow \frac{r^2}{rR}\left(r\frac{d^2R}{dr^2} + \frac{dR}{dr}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0 .$$

$$\Rightarrow \frac{1}{R}\left(r^2R'' + rR'\right) = -\frac{\Phi''}{\Phi} \qquad — (1)$$

Differentiating both sides with respect to r (or ϕ) equates to zero, so eqn (1) = constant = $m^2 > 0$. From (1)

$$r^2R'' + rR' - m^2R = 0 \qquad — (2)$$

and

$$\Phi'' + m^2\Phi = 0 \qquad — (3)$$

Equation (2) is in the form of an Euler - Cauchy equation.

$$\text{Let } R = r^\gamma$$

$$\therefore R' = \gamma . r^{\gamma-1} \quad \text{and} \quad R'' = \gamma.(\gamma-1) r^{-2}$$

$$\therefore (2) \Leftrightarrow r^2 \gamma(\gamma-1) r^{\gamma-2} + r \gamma . r^{\gamma-1} - m^2 r^\gamma = 0$$

$$\Rightarrow (\gamma.(\gamma-1) + \gamma - m^2) r^\gamma = 0$$

$$\Rightarrow \gamma^2 - m^2 = 0$$

$$\Rightarrow \gamma = \pm m. \quad \Rightarrow R(r) = E_m r^m + F_m r^{-m}$$

But this only provides one solution when $m=0$. We need a second solution. One way to do it is to return to equation (2) with $m=0$.

$$\text{Then} \quad r^2 R'' + r R' = 0. \qquad -(4)$$

$$\text{Let } q = R'$$

$$\Rightarrow r^2 q' + rq = 0.$$

$$\Rightarrow q' + \frac{1}{r} q = 0. \qquad -(5)$$

$$\text{Let } I(r) = \exp\left[\int \frac{dr}{r}\right]$$

$$= \exp\left[\ln(r)\right]$$

$$= r.$$

Then (5) becomes $\quad rq' + q = 0$

$$\Rightarrow \frac{d}{dr}\left[rq\right] = 0$$

$$\Rightarrow \int d\left[rq\right] = \int 0 \, dr$$

$$\Rightarrow rq = k_1 \qquad k_1 \in \mathbb{R}.$$

$$\Rightarrow r\frac{dR}{dr} = k_1$$

$$\Rightarrow dR = \frac{k_1}{r}\, dr$$

$$\Rightarrow \int dR = k_1 \int \frac{dr}{r}$$

$$\Rightarrow R = k_1 \ln(r) + k_2 .$$

$$k_2 \in \mathbb{R}.$$

We now have

$$R(r) = \begin{cases} k_1 \ln|r| + k_2 & m = 0 \\ E_m r^m + F_m r^{-m} & m > 0 \end{cases}$$

and

$$\Phi(\phi) = \begin{cases} A_0 + B_0 \phi & m = 0 \\ A_m \cos(m\phi) + B_m \sin(m\phi) & m > 0 \end{cases}$$

$$\therefore u(r,\phi) = (k_1 + k_2 \ln|r|)(A_0 + B_0\,\phi) + \sum_{m=1}^{\infty} \left(A_m\cos(m\phi) + B_m\sin(m\phi)\right)\left(E_m r^m + F_m r^{-m}\right)$$

2π periodicity is required so we set $B_0 = 0$ and let k_1 and k_2 absorb A_0.

BC1. $u(c,\phi) = 0$

$$\Rightarrow (k_1 + k_2 \ln|c|) + \sum_{m=1}^{\infty} \left(A_m\cos(m\phi) + B_m\sin(m\phi)\right)\left(E_m c^m + F_m c^{-m}\right)$$

$$\Rightarrow k_1 = -k_2 \ln|c|$$

and $E_m c^m = -F_m c^{-m} \Rightarrow F_m = -E_m c^{2m}$

$$\therefore u(r,\phi) = -k_2 \ln|c| + k_2 \ln|r| + \sum_{m=1}^{\infty} \left(A_m\cos(m\phi) + B_m\sin(m\phi)\right)\left(E_m r^m - E_m c^{2m} r^{-m}\right)$$

$$= k_2 \ln\left|\frac{r}{c}\right| + \sum_{m=1}^{\infty} \left(A_m\cos(m\phi) + B_m\sin(m\phi)\right) \cdot E_m c^m \left(\left(\frac{r}{c}\right)^m - \left(\frac{c}{r}\right)^m\right)$$

BC2 $u(b,\phi) = f(\phi) = A_0 \ln\left|\frac{b}{c}\right| + \sum_{m=1}^{\infty} \left(A_m\cos(m\phi) + B_m\sin(m\phi)\right)\left(\left(\frac{b}{c}\right)^m - \left(\frac{c}{b}\right)^m\right)$

where $A_0 = k_2$, A_m has absorbed $E_m c^m$, and B_m has also absorbed $E_m c^m$.

Now $A_0 \ln\left|\frac{b}{c}\right| = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\,d\phi$

$$\Rightarrow A_0 = \frac{1}{2\pi \ln\left|\frac{b}{c}\right|}\int_{-\pi}^{\pi} f(\phi)\,d\phi,$$

and $A_m\left(\left(\frac{b}{c}\right)^m - \left(\frac{c}{b}\right)^m\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\cos(m\phi)\,d\phi.$

$$\Rightarrow A_m = \frac{1}{\pi\left(\left(\frac{b}{c}\right)^m - \left(\frac{c}{b}\right)^m\right)}\int_{-\pi}^{\pi} f(\phi)\cos(m\phi)\,d\phi,$$

and $B_m\left(\left(\frac{b}{c}\right)^m - \left(\frac{c}{b}\right)^m\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\sin(m\phi)\,d\phi.$

Letting $\alpha_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\,d\phi$

$$\alpha_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\cos(m\phi)\,d\phi$$

$$\beta_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\sin(m\phi)\,d\phi.$$

$$A_0 = \frac{\alpha_0}{\ln\left|\frac{b}{c}\right|}, \quad A_m = \frac{\alpha_m}{\left(\left(\frac{b}{c}\right)^m - \left(\frac{c}{b}\right)^m\right)},$$

and $B_m = \dfrac{\beta_m}{\left(\left(\frac{b}{c}\right)^m - \left(\frac{c}{b}\right)^m\right)}$

$$\therefore u(r,\phi) = \alpha_0 \frac{\ln\left|\frac{r}{c}\right|}{\ln\left|\frac{b}{c}\right|} +$$

$$\sum_{m=1}^{\infty} \left(\frac{(r/c)^m - (c/r)^m}{(b/c)^m - (c/b)^m} \right) \left(\alpha_m \cos(m\phi) + \beta_m \sin(m\phi) \right)$$