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UNIVERSITY OF TASMANIA

EXAMINATIONS FOR DEGREES AND DIPLOMAS

November 2010

KMA354 Partial Differential Equations Applications & Methods 3

First and Only Paper

Examiner: Dr Michael Brideson

Time Allowed: TWO (2) hours.

Instructions:

- Attempt all FIVE (5) questions.
- All questions carry the same number of marks.

1. (a) Explain how and why the tangent plane can be used to solve a first order quasilinear partial differentiation equation.

(b) Solve the following Cauchy problem using the Method of Characterisitics.

$$\frac{\partial U}{\partial x} + (1+y)\frac{\partial U}{\partial y} = x\,,$$

$$U(x,0) = \sin((1+x)^2).$$

(c) A Cauchy problem is solved using the Method of Charateristics to give $U(x,y)$. Explain the relationship between contours of U and the characterstics.

2. Consider the nonhomogeneous wave equation

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} = -1\,, \qquad 0 < x < L, \quad t > 0\,;$$

with initial and boundary conditions,

$$\begin{aligned}
\text{BCs}: \quad & U(0,t) = 0\,, && t > 0\,; \\
& U(L,t) = 0\,, && t > 0\,; \\
\text{ICs}: \quad & U(x,0) = 1 + \frac{x}{2}(x - L)\,, && 0 < x < L\,; \\
& \frac{\partial U}{\partial t}(x,0) = 0 && 0 < x < L.
\end{aligned}$$

Make the substitution $U(x,t) = v(x,t) + \psi(x)$ and show the solution to be

$$U(x,t) = \frac{x}{2}(x - L) + \frac{4}{\pi}\sum_{n=0}^{\infty}\left(\frac{1}{2n+1}\right)\sin\left(\frac{(2n+1)\,\pi\,x}{L}\right)\cos\left(\frac{(2n+1)\,\pi\,t}{L}\right)\,.$$

3. Consider the Cauchy-Euler equation with $k \in \mathbb{R}$,

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - k^2 y = 0 .$$

(a) Show that $x = 0$ is a regular singular point.

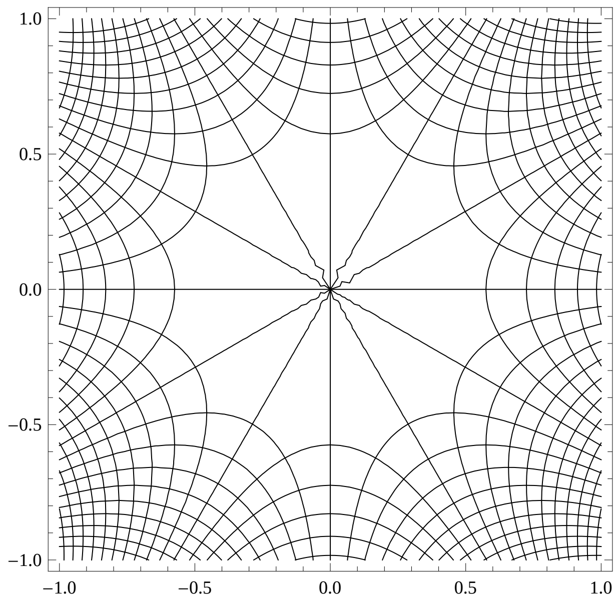
(b) Use Frobenius' method at at $x_0 = 0$ to confirm the general solution

$$y = A_1 x^k + A_2 x^{-k} .$$

(c) Under what conditions are the two solutions above linearly independent?

4. The plot below gives contours of the real and imaginary components of

$$\phi(z) \;=\; z^3 \;=\; (re^{i\phi})^3\,.$$



(a) Show this function to be analytic.

(b) Show that both the real and imaginary components are solutions to Laplace's equation.

(c) A potential function has the value zero long the lines $y = 0$ and $y = +\sqrt{3}\,x$. Along the line $y(3x^2 - y^2) \;=\; 10$ the potential has value 60. Modify the complex potential above and obtain a solution to Laplace's equation between the 3 lines.

Continued . . .

5. Consider a string fixed at its endpoints, $x = -\pi/2$ and $x = \pi/2$ with initial displacement $U(x, 0) = \cos(x)$ and zero initial velocity. All waves propagate with unit wave speed.

(a) Solve the problem using separation of variables. Begin by rescaling the x axis to aid the solution process.

(b) On an xt diagram indicate the 'domain of dependence' and 'range of influence' for the point $(0.25, \pi)$.

END OF EXAM PAPER

1(a) Consider a function $u(x,t)$ that is a smooth solution to the quasilinear PDE

$$a(x,t,u)\frac{\partial u}{\partial x} + b(x,t,u)\frac{\partial u}{\partial t} = f(x,t,u). \quad -①$$

At any point on the solution surface, eg $(x_0, t_0, u_0 = u(x_0,t_0))$, we can construct a Taylor series approximation, which to first order gives a tangent plane:

$$u(x,t) = u(x_0,t_0) + \frac{\partial u}{\partial x}(x_0,t_0)(x-x_0) + \frac{\partial u}{\partial t}(x_0,t_0)(t-t_0)$$

which can be modified into the total differential in its approximate form

$$\Delta u = \frac{\partial u}{\partial x}(x_0,t_0)\,\Delta x + \frac{\partial u}{\partial t}(x_0,t_0)\,\Delta t$$

where
$$\Delta u = u(x,t) = u(x_0,t_0).$$
$$\Delta x = x - x_0.$$
$$\Delta t = t - t_0.$$

Now, suppose that x and t are parameterised such that $x \equiv x(s)$ and $y \equiv y(s)$. We can construct a total derivative with respect to s:

$$\lim_{\Delta s \to 0}\frac{\Delta u}{\Delta s} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial t}\frac{dt}{ds} = \frac{du}{ds} \quad -②$$

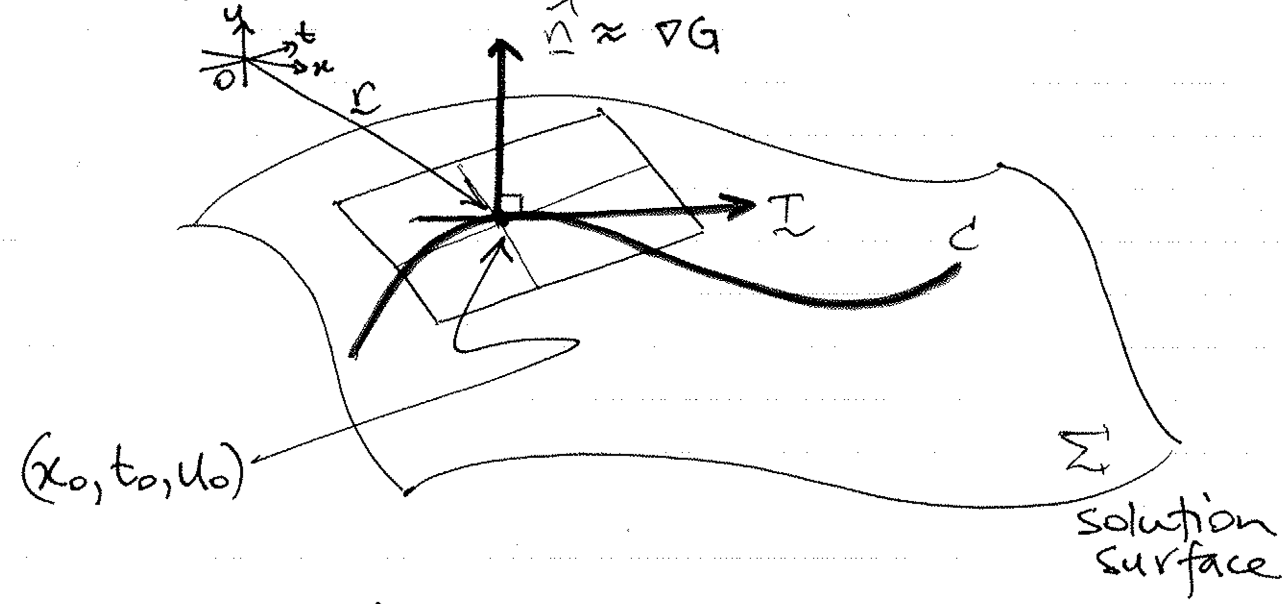
Equation ② now links the total derivative to the tangent plane to the solution surface and therefore to the PDE, equation ①.

Comparing terms in equations ① and ②,

$$a(x,t,u) = \frac{dx}{ds}, \quad b(x,t,u) = \frac{dt}{ds}, \text{ and } f(x,t,u) = \frac{du}{ds}.$$

subsidiary / complementary equations

$\{(x(s), t(s), u(s)) : s \in \mathbb{R}\}$ maps out a space curve C on the solution surface Σ.



A rearrangement of equation 2 yields

$$\left(\frac{dx}{ds}\frac{\partial u}{\partial x} + \frac{dt}{ds}\frac{\partial u}{\partial t} + (-1)\frac{du}{ds} \right) = 0$$

$$\Rightarrow \left(\frac{dx}{ds}\underline{\hat{e}_x} + \frac{dt}{ds}\underline{\hat{e}_t} + \frac{du}{ds}\underline{\hat{e}_u} \right) \cdot \left(\frac{\partial u}{\partial x}\underline{\hat{e}_x} + \frac{\partial u}{\partial t}\underline{\hat{e}_t} + (-1)\underline{\hat{e}_u} \right) = 0 \quad \text{—③}$$

Rewriting the solution surface $u = u(x,t)$ as a function of 3 variables

$$G(x,t,u) = u(x,t) - u = 0,$$

then

$$\nabla G = \frac{\partial G}{\partial x}\underline{\hat{e}_x} + \frac{\partial G}{\partial t}\underline{\hat{e}_t} + \frac{\partial G}{\partial u}\underline{\hat{e}_u}$$
$$= \frac{\partial u}{\partial x}\underline{\hat{e}_x} + \frac{\partial u}{\partial t}\underline{\hat{e}_t} + (-1)\underline{\hat{e}_u}.$$

Equation 3 becomes

$$\left(\frac{dx}{ds}\underline{\hat{e}_x} + \frac{dt}{ds}\underline{\hat{e}_t} + \frac{du}{ds}\underline{\hat{e}_u} \right) \cdot \nabla G = 0$$

A 3 dimensional gradient vector is always normal to the contour surface, so

$$\underset{\sim}{T} = \frac{dx}{ds}\,\underline{\hat{e_x}} + \frac{dt}{ds}\,\underline{\hat{e_t}} + \frac{du}{ds}\,\underline{\hat{e_u}}$$

must be tangent to the surface and therefore reside in the tangent plane.

Finally, if the space curve C is mapped out by the parametric position vector

$$\underline{r}(s) = x(s)\,\underline{\hat{e_x}} + t(s)\,\underline{\hat{e_t}} + u(s)\,\underline{\hat{e_u}}$$

then the parametric velocity vector, which gives the trajectory of the position vector, is

$$\begin{aligned}\underset{\sim}{v}(s) &= \frac{dr}{ds}\\ &= \frac{dx}{ds}\,\underline{\hat{e_x}} + \frac{dt}{ds}\,\underline{\hat{e_t}} + \frac{du}{ds}\,\underline{\hat{e_u}}\\ &= \underset{\sim}{T}.\end{aligned}$$

Thus the subsidiary/complementary equations describe the evolution of a solution curve C on the solution surface Σ.

b) $\frac{\partial u}{\partial x} + (1+y)\frac{\partial u}{\partial y} = x$ — ①

$u(x,0) = \sin((1+x)^2)$

Consider the total derivative

$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds}$ — ②

Comparing ① and ②,

$\frac{dx}{ds} = 1$, $\frac{dy}{ds} = 1+y$, $\frac{du}{ds} = x$.

Now $ds = \frac{dx}{1} = \frac{dy}{1+y} = \frac{du}{x}$,

giving $\int dx = \int \frac{dy}{1+y}$

$\Rightarrow x + c_1 = \ln(1+y)$, $c_1 \in \mathbb{R}$.

$\Rightarrow 1+y = e^{x+c_1}$

$\Rightarrow k = e^{-x}(1+y)$, $k = e^{c_1}$.

Also, $\frac{dx}{1} = \frac{du}{x}$

$\Rightarrow \int x\,dx = \int du$

$\Rightarrow \frac{x^2}{2} + c_2 = u$, $c_2 \in \mathbb{R}$

$\therefore u = \frac{x^2}{2} + F(e^{-x}(1+y))$

Initial condition : $u(x,0) = \sin((1+x)^2)$

$$\therefore \frac{x^2}{2} + F(e^{-x}) = \sin((1+x)^2)$$

$$\Rightarrow F(e^{-x}) = \sin((1+x)^2) - \frac{x^2}{2}$$

$$\Rightarrow F(w) = \sin((1-\ln(w))^2) - \frac{(\ln(w))^2}{2}$$

$$\Rightarrow F(e^{-x}(1+y)) = \sin((1-\ln(e^{-x}(1+y)))^2) - \frac{(\ln(e^{-x}(1+y)))^2}{2}$$

$$= \sin((1+x-\ln(1+y))^2) - \frac{(-x+\ln(1+y))^2}{2}$$

$$\therefore u(x,y) = \frac{x^2}{2} + \sin((1+x-\ln(1+y))^2)$$

$$- \frac{(\ln(1+y)-x)^2}{2} \quad , \quad y > -1.$$

or some equivalent form.

2. $\dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = -1 \qquad x \in (0,L), \; t > 0$

BC1: $u(0,t) = 0, \quad t > 0;$

BC2: $u(L,t) = 0, \quad t > 0;$

IC1: $u(x,0) = 1 + \dfrac{x}{2}(x-L), \quad x \in (0,L);$

IC2: $\dfrac{\partial u}{\partial t}(x,0) = 0, \quad x \in (0,L).$

Let $u(x,t) = v(x,t) + \psi(x)$

Then $\dfrac{\partial u}{\partial t} = \dfrac{\partial v}{\partial t}, \quad \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 v}{\partial t^2},$

$\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial x} + \psi', \quad \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 v}{\partial x^2} + \psi''.$

So the pde becomes

$$\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} - \psi'' = -1,$$

and the boundary and initial conditions become

BC1: $u(0,t) = v(0,t) + \psi(0) = 0$

BC2: $u(L,t) = v(L,t) + \psi(L) = 0$

IC1: $u(x,0) = v(x,0) + \psi(x) = 1 + \dfrac{x}{2}(x-L)$

IC2: $\dfrac{\partial u}{\partial t}(x,0) = \dfrac{\partial v}{\partial t}(x,0) = 0.$

We now separate into two subproblems by letting

$$\psi'' = 1 \;, \quad \psi(0) = 0 \;, \text{ and } \psi(L) = 0 \;.$$

This leaves

pde$_2$: $\quad \dfrac{\partial^2 v}{\partial t^2} - \dfrac{\partial^2 v}{\partial x^2} = 0 \,,$

BC1$_2$: $\quad v(0,t) = 0,$
BC2$_2$: $\quad v(L,t) = 0,$
IC1$_2$: $\quad v(x,0) = 1 + \dfrac{x}{2}(x-L) - \psi(x)$

IC2$_2$: $\quad \dfrac{\partial v}{\partial t}(x,0) = 0.$

Subproblem 1.

$$\psi'' = 1$$
$$\Rightarrow \psi' = x + c_1$$
$$\Rightarrow \psi = \frac{x^2}{2} + c_1 x + c_2 \qquad c_1, c_2 \in \mathbb{R}.$$

$\psi(0) = 0 \quad \Rightarrow \quad c_2 = 0$

$$\therefore \psi = \frac{x^2}{2} + c_1 x$$

$\psi(L) = 0 \quad \Rightarrow \quad \dfrac{L^2}{2} + c_1 L = 0$

$$\Rightarrow c_1 = -\frac{L}{2}$$

$$\therefore \psi(x) = \frac{x}{2}(x - L).$$

Subproblem 2.

pde$_2$: $\quad \dfrac{\partial^2 v}{\partial t^2} - \dfrac{\partial^2 v}{\partial x^2} = 0.$

Let $v(x,t) = X(x)\,T(t)$

$$\Rightarrow \frac{\partial^2 v}{\partial t^2} = XT''$$

and $$\frac{\partial^2 v}{\partial x^2} = X''T.$$

$\therefore$ pde$_2$ becomes $XT'' - X''T = 0$

$$\Rightarrow \frac{T''}{T} = \frac{X''}{X}.$$

This equation must equal a constant since differentiating both sides with respect to x (or t) will give zero.

With both boundary conditions equalling zero we choose the constant be be $-\lambda^2 < 0$.

$$\therefore \frac{T''}{T} = \frac{X''}{X} = -\lambda^2$$

$$\Rightarrow \quad T'' + \lambda^2 T = 0$$

and $X'' + \lambda^2 X = 0$,

with respective solutions

$$T(t) = a_1 \cos(\lambda t) + a_2 \sin(\lambda t)$$
$$X(x) = a_3 \cos(\lambda x) + a_4 \sin(\lambda x).$$

BC1$_2$: $v(0,t) = X(0)T(t) = 0$

$\Rightarrow X(0) = 0$ since $T(t) \neq 0 \ \forall t$.

$\Rightarrow \quad a_3 = 0$.

$\therefore \ X(x) = a_4 \sin(\lambda x)$.

BC2$_2$: $v(L,t) = X(L)T(t) = 0$

$\Rightarrow X(L) = 0$ since $T(t) \neq 0 \ \forall t$.

$\Rightarrow \ a_4 \sin(\lambda L) = 0$.

X will be trivial if $a_4 = 0$, so we set

$\lambda L = n\pi \quad n = 1, 2, 3, \ldots$

$\Rightarrow \ \lambda = \frac{n\pi}{L}$.

$$\therefore \ X_n(x) = a_{4_n} \sin\left(\frac{n\pi x}{L}\right)$$

IC2$_2$: $\frac{\partial v}{\partial t}(x,0) = X(x)T'(0) = 0$

$\Rightarrow T'(0) = 0$ since $X(x) \neq 0 \ \forall x$.

$$T'(t) = \lambda(-a_1 \sin(\lambda t) + a_2 \cos(\lambda t))$$
$$\therefore T'(0) = \lambda a_2 = 0$$
$$\lambda \neq 0 \quad \therefore a_2 = 0.$$

$$\therefore T(t) = a_1 \cos(\lambda t)$$
and $T_n(t) = a_{1n} \cos\left(\frac{n\pi t}{L}\right)$

$$\therefore v_n(x,t) = \alpha_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi t}{L}\right)$$
where $\alpha_n = a_{1n} a_{4n}$.

$IC1_2$: $v(x,0) = X(x)T(0)$

$$= 1 + \frac{x}{2}(x-L) - \psi(x)$$
$$= 1 + \frac{x}{2}(x-L) - \frac{x}{2}(x-L)$$
$$= 1.$$

$$\therefore v(x,0) = \sum_{n=1}^{\infty} v_n(x,0) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi x}{L}\right) = 1.$$

Using orthogonality conditions,

$$\alpha_n = \frac{1}{L/2} \int_0^L \sin\left(\frac{n\pi x}{L}\right) dx$$
$$= \frac{2}{L} \left[\frac{-L}{n\pi} \cos\left(\frac{n\pi x}{L}\right)\right]_0^L$$
$$= \frac{-2}{n\pi} \left[\cos(n\pi) - 1\right]$$
$$= \frac{4}{n\pi} \quad \text{for odd } n.$$

$$\therefore u(x,t) = \frac{x}{2}(x-L) + \sum_{n=1,3,5}^{\infty} \frac{4}{n\pi} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi t}{L}\right)$$

$$= \frac{x}{2}(x-L) + \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)} \sin\left(\frac{(2n+1)\pi x}{L}\right) \cos\left(\frac{(2n+1)\pi t}{L}\right)$$

3. $$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-k^2y=0 \qquad \text{— (1)}$$

(a) $$\Rightarrow \frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}-\frac{k^2}{x^2}y=0.$$

Let $p(x)=\frac{1}{x}$ and $q(x)=\frac{-k^2}{x^2}$.

Let $x_0=0$.

$$\lim_{x\to x_0}(x-x_0)p(x)=\lim_{x\to 0}x\left(\frac{1}{x}\right)=1$$

$$\lim_{x\to x_0}(x-x_0)^2q(x)=\lim_{x\to 0}x^2\left(\frac{-k^2}{x^2}\right)=-k^2$$

Since both limits are finite, $x_0=0$ is a regular singular point.

(b) Let $y=\sum_{m=0}^{\infty}a_mx^{m+r}$, $a_0\neq 0$.

$$\therefore \frac{dy}{dx}=\sum_{m=0}^{\infty}a_m(m+r)x^{m+r-1}$$

and $$\frac{d^2y}{dx^2}=\sum_{m=0}^{\infty}a_m(m+r)(m+r-1)x^{m+r-2}.$$

Substituting these expressions into (1) gives

$$x^2\sum_{m=0}^{\infty}a_m(m+r)(m+r-1)x^{m+r-2}$$
$$+x\sum_{m=0}^{\infty}a_m(m+r)x^{m+r-1}-k^2\sum_{m=0}^{\infty}a_mx^{m+r}=0$$

$$\Rightarrow \sum_{m=0}^{\infty}a_m(m+r)(m+r-1)x^{m+r}+\sum_{m=0}^{\infty}a_m(m+r)x^{m+r}$$
$$-k^2\sum_{m=0}^{\infty}a_mx^{m+r}=0$$

$$\Rightarrow \sum_{m=0}^{\infty}\left((m+r)(m+r-1)+(m+r)-k^2\right)a_mx^{m+r}=0$$

$$\Rightarrow \sum_{m=0}^{\infty} \left((m+r)^2 - k^2\right) a_m x^{m+r} = 0 \qquad — ②$$

This is satisfied on the interval of convergence if

$$\left((m+r)^2 - k^2\right) a_m = 0 \qquad \forall\ m.$$

When $m=0$ we obtain $r^2 - k^2 = 0$ (indicial equation) since $a_0 \neq 0$.

$$\therefore\ r = \pm k.$$

$$\Rightarrow r_1 = +k \quad \text{and} \quad r_2 = -k, \quad r_1 > r_2.$$

Solution 1: $r_1 = +k$

substituting $r_1 = k$ into ② gives

$$\sum_{m=0}^{\infty} (m^2 + 2mk) a_m x^{m+k} = 0$$

$$\Rightarrow 0 a_0 x^k + \sum_{m=1}^{\infty} (m+2k) m a_m x^{m+k} = 0$$

$$\Rightarrow a_m = 0 \quad \text{for } m \geq 1 \quad \text{since } m+2k \neq 0.$$

$$\therefore\ y_1 = \sum_{m=0}^{\infty} a_m x^{m+k} = a_0 x^k. \qquad — ③$$

Solution 2: $r_2 = -k$

substituting $r_2 = -k$ into ② gives

$$\sum_{m=0}^{\infty} (m^2 - 2mk) a_m x^{m-k} = 0$$

$$\Rightarrow 0 a_0 x^{-k} + \sum_{m=1}^{\infty} (m-2k) m a_m x^{m-k} = 0$$

$\Rightarrow a_m = 0$ for $m \geq 1$ unless $m = 2k$; ie. k an integer.

With $k \notin \mathbb{Z}$ $\quad y_2 = \sum_{m=0}^{\infty} a_m x^{m-k} = a_0 x^{-k} \qquad — ④$

Combining ③ and ④, $y = A_1 x^k + A_2 x^{-k}$.

(c) The two solutions are linearly independent if the Wronskian does not equate to zero.

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} x^k & x^{-k} \\ kx^{k-1} & -kx^{-k-1} \end{vmatrix}$$

$$= x^k(-kx^{-k-1}) - kx^{k-1}x^{-k}$$

$$= -kx^{-1} - kx^{-1}$$

$$= -2kx^{-1}$$

$$\neq 0 \text{ if } k \neq 0.$$

4. (a) $$\Phi(z) = z^3 = (re^{i\phi})^3$$
$$= (x+iy)^3$$
$$= x^3 + 3x^2 iy + 3x(iy)^2 + (iy)^3$$
$$= (x^3 - 3xy^2) + i(3x^2y - y^3). \quad \text{—} (*)$$

Now $\Phi(z) \equiv \Phi(x,y) = u(x,y) + i\,v(x,y)$.

$$\therefore u(x,y) = x^3 - 3xy^2 = x(x^2 - 3y^2)$$
and $$v(x,y) = 3x^2y - y^3 = y(3x^2 - y^2)$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \quad \frac{\partial u}{\partial y} = -6xy.$$
$$\frac{\partial v}{\partial x} = 6xy, \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2.$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence $\Phi(z)$ is analytic as the Cauchy-Riemann relations are satisfied.

(b). $$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$
$$= \frac{\partial}{\partial x}(3x^2 - 3y^2) + \frac{\partial}{\partial y}(-6xy)$$
$$= 6x - 6x$$
$$= 0.$$

$$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$$
$$= \frac{\partial}{\partial x}(6xy) + \frac{\partial}{\partial y}(3x^2 - 3y^2)$$
$$= 6y - 6y$$
$$= 0.$$

(c) With $v(x,y) = y(3x^2-y^2)$,

when $y=0$, $v=0$;

$y=\sqrt{3}x$, $v=\sqrt{3}x(3x^2-(\sqrt{3}x)^2)$
$= 0$;

and when $y(3x^2-y^2)=10$, $v=10$.

But we require $v=60$ when $y(3x^2-y^2)=10$.

Let $v(x,y) = 6y(3x^2-y^2)$

$\therefore v(x,0) = 0$
$v(x,\sqrt{3}x) = 0$
and $v(x,y) = 60$ when $y(3x^2-y^2)=10$.

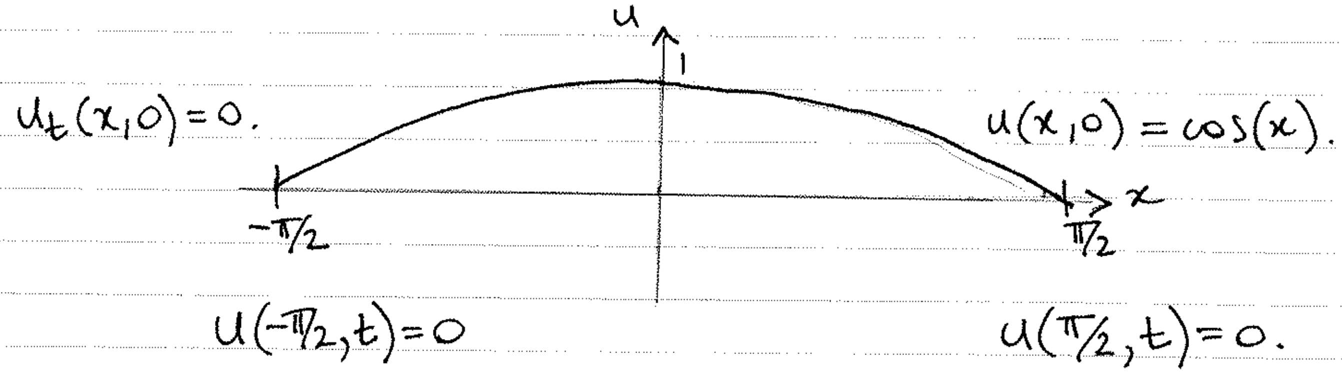
Now Φ must be modified accordingly.

Let $\Phi(z) = 6z^3$
$= 6(x+iy)^3$
$= 6((x^3-3xy^2)+i(3x^2y-y^3))$ from ✱

giving $u(x,y) = 6x(x^2-3y^2)$

and $v(x,y) = 6y(3x^2-y^2)$ as required.

5.



Rescaling, let $x^* = x + \frac{\pi}{2}.$

$$\Rightarrow x = x^* - \frac{\pi}{2}$$

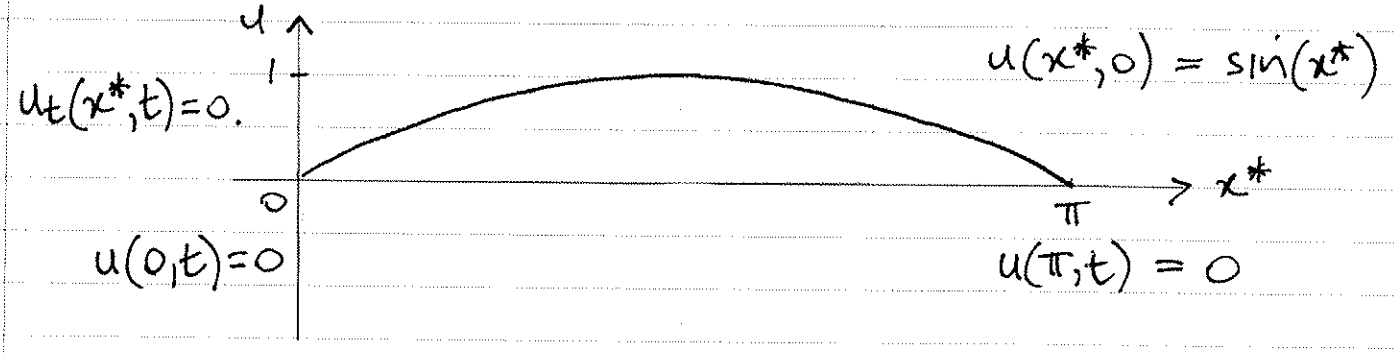
In terms of x^* and t, the boundary conditions and initial conditions become

$$u(-\pi/2+\pi/2,t) = u(0,t) = 0.$$

$$u(\pi/2+\pi/2,t) = u(\pi,t) = 0.$$

$$u_t(x^*,0) = 0$$

$$\begin{aligned} u(x^*,0) &= \cos(x^*-\pi/2) \\ &= \cos(x^*)\cos(\pi/2) + \sin(x^*)\sin(\pi/2) \\ &= \sin(x^*). \end{aligned}$$



The governing equation is

$$\frac{\partial^2 u}{\partial x^{*2}} = \frac{\partial^2 u}{\partial t^2}$$

Let $u(x^*,t) = X(x^*)T(t)$,

then $\dfrac{\partial^2 u}{\partial x^{*2}} = X''(x)\,T(t)$

and $\frac{\partial^2 u}{\partial t^2} = X(x)\, T''(t)$.

The pde becomes $X'' T = X T''$

$$\Rightarrow \frac{X''}{X} = \frac{T''}{T}.$$

Differentiating both sides with respect to x^* (or t) must equate to zero, therefore

$$\frac{X''}{X} = \frac{T''}{T} = m, \text{ a constant.}$$

Based on the form of the boundary conditions, let $m = -\lambda^2 < 0$.

$$\therefore X'' + \lambda^2 X = 0$$
and
$$T'' + \lambda^2 T = 0.$$

with respective solutions

$$X(x^*) = a_1 \cos(\lambda x^*) + a_2 \sin(\lambda x^*)$$
$$T(t) = a_3 \cos(\lambda t) + a_4 \sin(\lambda t).$$

$$u(0,t) = X(0)\,T(t) = 0$$
$$\Rightarrow X(0) = 0 \quad \text{since } T(t) \neq 0 \ \forall t.$$
$$\therefore a_1 = 0.$$
So $X(x^*) = a_2 \sin(\lambda x^*)$.

$$u(\pi,t) = X(\pi)\,T(t) = 0$$
$$\Rightarrow X(\pi) = 0 \quad \text{since } T(t) \neq 0 \ \forall t.$$
$a_2 \neq 0 \quad \therefore \sin(\lambda\pi) = 0$
$$\Rightarrow \lambda \in \mathbb{Z}^+.$$

$$\therefore X_\lambda(x^*) = a_{2\lambda} \sin(\lambda x^*).$$

$$\frac{\partial u}{\partial t} = X(x^*)\, T'(t).$$

$$T'(t) = \lambda(-a_3 \sin(\lambda t) + a_4 \cos(\lambda t))$$

$$\frac{\partial u}{\partial t}(x^*,0) = X(x^*)\,T'(0) = 0$$

$$\Rightarrow T'(0) = 0 \quad \text{since } X(x^*) \neq 0 \ \forall x^*.$$

$$T'(0) = 0$$
$$\Rightarrow \lambda a_4 = 0$$
$$\Rightarrow a_4 = 0 \quad \text{since } \lambda \neq 0.$$

$$\therefore T_\lambda(t) = a_{3\lambda} \cos(\lambda t)$$

Now $u(x^*, t) = \sum_{\lambda=1}^{\infty} X_\lambda(x^*) T_\lambda(t)$

$$= \sum_{\lambda=1}^{\infty} a_\lambda \sin(\lambda x^*) \cos(\lambda t)$$

where $a_\lambda = a_{2\lambda} a_{3\lambda}$.

$$u(x^*, 0) = \sum_{\lambda=1}^{\infty} a_\lambda \sin(\lambda x^*) = \sin(x^*).$$

It is immediately obvious that

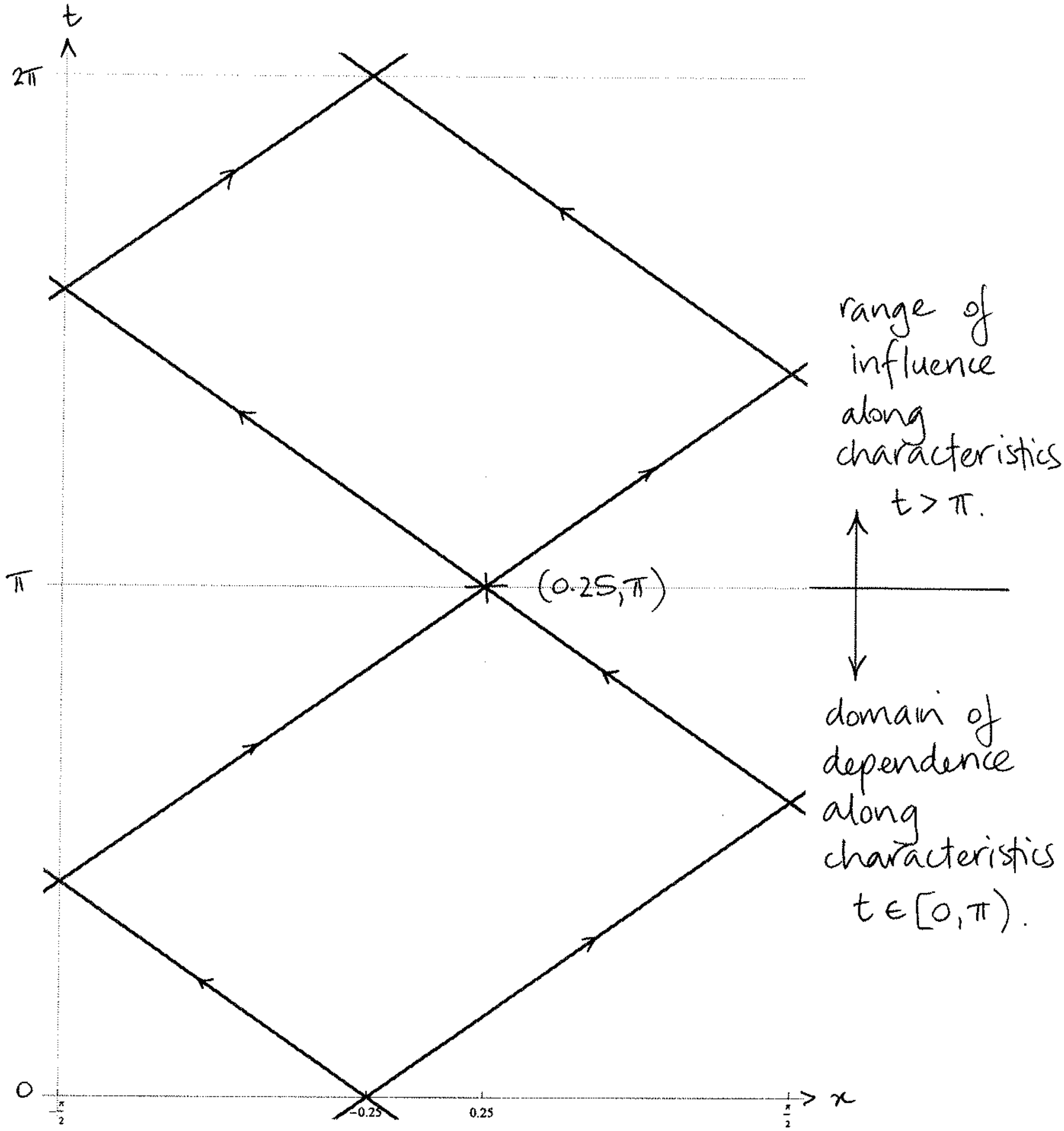
$$a_\lambda = \begin{cases} 0 & \lambda \neq 1 \\ 1 & \lambda = 1 \end{cases}$$

$$\therefore u(x^*, t) = \sin(x^*) \cos(t)$$

Transforming back to x and t.

$$u(x, t) = \sin\left(x + \frac{\pi}{2}\right) \cos(t)$$
$$= \left(\sin(x) \cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right) \cos(x)\right) \cos(t)$$
$$= \cos(x) \cos(t).$$

(b)


t
2π
π
0
x
−π/2
−0.25
0.25
π/2
(0.25, π)
range of influence along characteristics t > π.
domain of dependence along characteristics t ∈ [0, π).