2007 exam

$$\frac{d^2\Phi}{dx^2} + k^2\Phi = 0$$

$$\Phi = \sum_{m=0}^{\infty} a_m x^{m+r} \qquad a_0 \neq 0.$$

$$\therefore r(r-1)a_0x^{r-2} + (1+r)ra_1x^{r-1}$$
$$+ \sum_{m=0}^{\infty} (m+r+2)(m+r+1)a_{m+2}x^{m+r} + k^2x^{m+r}a_m$$
$$= 0.$$

$$\Rightarrow r(r-1)a_0 = 0 \qquad — ①$$

$$(1+r)ra_1 = 0 \qquad — ②$$

and $(m+r+2)(m+r+1)a_{m+2} + k^2a_m = 0$,

$$\Rightarrow a_{m+2} = \frac{-k^2a_m}{(m+r+2)(m+r+1)}. \qquad — ③$$

From ① $\quad r = 0$ or $r-1 = 0$

$\therefore r = 0, 1.$

From ②, with $r=0$ $\quad a_1 \neq 0$.

with $r=1$ $\quad a_1 = 0$. — ④

From ③, with $r=0$

$$a_{m+2} = \frac{-k^2a_m}{(m+2)(m+1)}. \qquad — ⑤$$

with $r=1$

$$a_{m+2} = \frac{-k^2a_m}{(m+3)(m+2)} \qquad — ⑥$$

$$\Phi = \sum_{m=0}^{\infty} a_m x^{m+r}$$

$\boxed{r=0}$, $$\Phi = \sum_{m=0}^{\infty} a_m x^m$$

$$= \sum_{m=0,2,4}^{\infty} a_m x^m + \sum_{m=1,3,5}^{\infty} a_m x^m.$$

Looking at the recurrence relationship (5),

$$a_{m+2} = \frac{-k^2 a_m}{(m+2)(m+1)}, \text{ for}$$

$m=0$, $a_2 = \dfrac{-k^2 a_0}{2\times 1}$;

$m=1$, $a_3 = \dfrac{-k^2 a_1}{3\times 2}$;

$m=2$, $a_4 = \dfrac{-k^2 a_2}{4\times 3} = \dfrac{+k^4 a_0}{4\times 3\times 2\times 1}$;

$m=3$, $a_5 = \dfrac{-k^2 a_3}{5\times 4} = \dfrac{+k^4 a_1}{5\times 4\times 3\times 2}$;

$m=4$, $a_6 = \dfrac{-k^2 a_4}{6\times 5} = -\dfrac{k^6 a_0}{6!}$;

$m=5$, $a_7 = \dfrac{-k^2 a_5}{7\times 6} = -\dfrac{k^6 a_1}{7!}$.

In general,

$$a_m = (-1)^{\frac{m}{2}} \frac{k^m}{m!} a_0 \qquad \text{even } m$$

$$a_m = (-1)^{\frac{m-1}{2}} \frac{k^{m-1}}{m!} a_1 \qquad \text{odd } m.$$

$$= (-1)^{\frac{m-1}{2}} \frac{k^m}{km!} a_1$$

$$\therefore \Phi = a_0 \sum_{m=0,2,4}^{\infty} (-1)^{\frac{m}{2}} \frac{k^m x^m}{m!} + a_1 \sum_{m=1,3,5}^{\infty} (-1)^{\frac{m-1}{2}} \frac{k^m}{km!} x^m$$

$$= a_0 \sum_{n=0}^{\infty} (-1)^n \frac{(kx)^{2n}}{(2n)!} + \frac{a_1}{k} \sum_{n=0}^{\infty} (-1)^n \frac{(kx)^{2n+1}}{(2n+1)!}$$

$$= A_0 \cos(kx) + B_0 \sin(kx)$$

$\boxed{r=1}$

$$\Phi = \sum_{m=0}^{\infty} a_m x^{m+1}$$

$$= \sum_{m=0,2,4}^{\infty} a_m x^{m+1} + \sum_{m=1,3,5,}^{\infty} a_m x^{m+1}$$

Looking at the recurrence relationship (6),

$$a_{m+2} = \frac{-k^2 a_m}{(m+3)(m+2)},$$

all of the odd coefficients will disappear because of (4): $a_1 = 0$.

$m=0$, $\quad a_2 = \dfrac{-k^2 a_0}{3\times 2}$;

$m=2$, $\quad a_4 = \dfrac{-k^2 a_2}{5\times 4} = \dfrac{k^4 a_0}{5!}$

$m=4$, $\quad a_6 = \dfrac{-k^2 a_4}{7\times 6} = \dfrac{-k^6 a_0}{7!}$

$\therefore$ In general,

$$a_m = \frac{(-1)^{\frac{m}{2}} a_0 k^m}{(m+1)!}$$

$$\therefore \quad \Phi = a_0 \sum_{m=0,2,4}^{\infty} (-1)^{m/2} \frac{k^m x^{m+1}}{(m+1)!}$$

$$= \frac{a_0}{k} \sum_{m=0,2,4}^{\infty} (-1)^{m/2} \frac{(kx)^{m+1}}{(m+1)!}$$

$$= \frac{a_0}{k} \sum_{n=0}^{\infty} (-1)^n \frac{(kx)^{2n+1}}{(2n+1)!}$$

$$= A_1 \sin(kx).$$