Pages: 6

Student ID No: ________________ Questions : 5

UNIVERSITY OF TASMANIA

EXAMINATIONS FOR DEGREES AND DIPLOMAS

October / November 2011

KMA354 Partial Differential Equations Applications & Methods

First and Only Paper

Examiner: Dr Michael Brideson

Time Allowed: TWO (2) hours.

Instructions:

- Attempt all FIVE (5) questions.
- All questions carry the same number of marks.

1. A solution to the Telegraph equation,

$$\frac{\partial^2 E}{\partial t^2} + \alpha \frac{\partial E}{\partial t} - c^2 \frac{\partial^2 E}{\partial x^2} = 0\,,$$

can be obtained by letting $E(x,t) = v(t)\,U(x,t)$ with a view to removing the first order time derivative.

Use this process to obtain the Klein-Gordon equation,

$$\frac{\partial^2 U}{\partial t^2} - c^2 \frac{\partial^2 U}{\partial x^2} - \frac{\alpha^2}{4} U = 0\,.$$

Continued ...

2. (a) Use the Method of Characteristics to solve the following initial value problem.

$$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial t} + 2U = 0\,,$$
$$U(x,0) = \sin(x).$$

(b) Use the one sided Green's function technique to solve

$$\begin{aligned} U''(x) + U'(x) &= e^x \qquad x > 0 \\ U(0) &= 1 \\ U'(0) &= 0. \end{aligned}$$

Continued ...

3. Consider the nonhomogeneous wave equation

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} = x\,, \qquad 0 < x < 1, \quad t > 0\,;$$

with initial and boundary conditions,

$$\begin{aligned} \text{BCs :} \quad & U(0,t) \;=\; 0\,, && t > 0\,; \\ & U(1,t) \;=\; 0\,, && t > 0\,; \\ \text{ICs :} \quad & U(x,0) \;=\; x\,, && 0 < x < 1\,; \\ & \frac{\partial U}{\partial t}(x,0) \;=\; 0 && 0 < x < 1. \end{aligned}$$

Make a suitable substitution to turn this into two subproblems and solve for $U(x,t)$.

You do not have to evaluate the integral for the Fourier coefficients but you must show its derivation.

Continued ...

4. (a) Use an appropriate power series method to solve

$$\frac{dy}{dx} - y = x^2 .$$

Hint: Consider the Maclaurin series for e^x.

(b) Consider the Cauchy-Euler equation,

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - k^2 y = 0 \qquad \text{with } k \in \mathbb{R}.$$

Show that $x = 0$ is a regular singular point.

5. (a) In the context of the regular Sturm-Liouville boundary value problem

$$\frac{d}{dx}\left[H(x)\,\frac{dy}{dx}\right] \;+\; (Q(x)+\lambda W(x))y \;=\; 0 \qquad -\infty < a \le x \le b < \infty$$

$$a_1\,y(a) + a_2\,y'(a) = 0$$

$$b_1\,y(b) + b_2\,y'(b) = 0\,,$$

interpret the expression

$$(\lambda_j - \lambda_k)\int_a^b W(x)\,y_j(x)\,y_k(x)\,dx \;=\; \Big[H(x)(y_j(x)\,y_k'(x) - y_k(x)\,y_j'(x))\Big]_a^b\,.$$

You do not have to consider all specific cases for the right hand side, but you must comment on the two general cases, $j = k$ and $j \neq k$.

(b) The equation,

$$(1-x^2)\,\frac{d^2y}{dx^2} \;-\; 2x\,\frac{dy}{dx} \;+\; l\,(l+1)\,y \;=\; 0\,,$$

has polynomial solutions $P_l(x)$ for integer l.

Put the equation in self-adjoint form and give an orthogonality relation.

END OF EXAM PAPER

Q1 $E_{tt} + \alpha E_t - c^2 E_{xx} = 0. \quad — ①$

$E = v(t)u(x,t).$

$E_t = v'u + vu_t$

$E_{tt} = v''u + v'\cancel{u_{\$}}u_t + v'u_t + vu_{tt}$
$= v''u + 2v'u_t + vu_{tt}.$

$E_{xx} = vu_{xx}.$

$\therefore$ ① becomes.

$v''u + 2v'u_t + vu_{tt} + \alpha(v'u + vu_t)$
$\quad - c^2 v u_{xx} = 0.$

$\Rightarrow v u_{tt} - c^2 v u_{xx} + v''u + \alpha v'u + (2v' + \alpha v)u_t$
$= 0. \quad — ②$

To remove the time derivative, u_t, we set $2v' + \alpha v = 0$

$\Rightarrow \dfrac{dv}{dt} = -\dfrac{\alpha v}{2}$

$\Rightarrow \displaystyle\int \frac{dv}{v} = \int -\frac{\alpha}{2}\, dt$

$\Rightarrow \ln v = -\dfrac{\alpha t}{2} + c_1$

$\Rightarrow v = k\exp\left(-\dfrac{\alpha t}{2}\right)$

Now ②: $v\left(u_{tt} - c^2 u_{xx}\right) + \left(v'' + \alpha v'\right)u = 0. \quad — ③$

$v' = -\dfrac{\alpha}{2} k\exp\left(-\dfrac{\alpha t}{2}\right) = -\dfrac{\alpha v}{2}$

$v'' = \dfrac{\alpha^2}{4} k\exp\left(-\dfrac{\alpha t}{2}\right) = \dfrac{\alpha^2}{4} v.$

$$\therefore \quad v'' + \alpha v = \frac{\alpha^2}{4} v - \frac{\alpha^2}{2} v$$
$$= -\frac{\alpha^2}{4} v.$$

Now (3)': $v\left(u_{tt} - c^2 u_{xx}\right) - \frac{\alpha^2}{4} v u = 0$

$$\Rightarrow \quad \left(u_{tt} - c^2 u_{xx} - \frac{\alpha^2}{4} u\right) v = 0.$$

Since $v \neq 0 \quad \forall t$

$$u_{tt} - c^2 u_{xx} - \frac{\alpha^2}{4} u = 0.$$

ie the Klein-Gordon equation.

2(a) $$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + 2u = 0$$
$$u(x,0) = \sin(x).$$

Using the method of characteristics we rewrite the PDE as
$$\frac{\partial u}{\partial x}(1) + \frac{\partial u}{\partial t}(1) = -2u$$
and compare it against the total derivative with respect to parameter s:
$$\frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial t}\frac{dt}{ds} = \frac{du}{ds}$$

Then $\frac{dx}{ds} = 1$, $\frac{dt}{ds} = 1$; $\frac{du}{ds} = -2u$.

Now, $\frac{dx}{ds} / \frac{dt}{ds} = \frac{dx}{dt} = 1$.

Integrating both sides with respect to t yields
$$x = t + k \qquad k \in \mathbb{R}.$$
$$\Rightarrow k = x - t.$$
This is the characteristic equation.

To solve for u we can pair $\frac{du}{ds}$ with $\frac{dx}{ds}$ or $\frac{dt}{ds}$. Working with $\frac{dx}{ds}$:
$$\frac{du}{ds} / \frac{dx}{ds} = \frac{du}{dx} = -2u$$
This can be solved using the integrating factor technique or as a separable equation.
$$\therefore \frac{1}{u}\frac{du}{dx} = -2$$
and after integrating both sides with respect to x we obtain
$$\ln|u| = -2x + c$$
$$\Rightarrow u = e^{-2x+c}$$
$$= Ae^{-2x} \qquad \text{where } A = e^c$$

but A is a function of the characteristics so

$$u = A(k)e^{-2x}$$
$$= A(x-t)e^{-2x}.$$

Using the initial condition $u(x,0) = \sin(x)$,

$$A(x)e^{-2x} = \sin(x)$$
$$\Rightarrow A(x) = e^{2x}\sin(x)$$
$$\Rightarrow A(x-t) = e^{2(x-t)}\sin(x-t)$$

$$\therefore u(x,t) = e^{2(x-t)}\sin(x-t)e^{-2x}$$
$$= e^{-2t}\sin(x-t).$$

Check:

$$\frac{\partial u}{\partial x} = e^{-2t}\cos(x-t)$$

$$\frac{\partial u}{\partial t} = -2e^{-2t}\sin(x-t) - e^{-2t}\cos(x-t)$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + 2u$$

$$= e^{-2t}\cos(x-t) - 2e^{-2t}\sin(x-t) - e^{-2t}\cos(x-t) + 2e^{-2t}\sin(x-t)$$

$$= 0 \checkmark$$

$\therefore$ the PDE is confirmed.

$$I(x,0) = e^{-0}\sin(x-0)$$
$$= \sin(x)$$

and the initial condition is confirmed.

(b) ODE : $u''(x) + u'(x) = e^x \qquad x > 0$

BC1 : $u(0) = 1$

BC2 : $u'(0) = 0$

The ODE is in the form
$$u'' + h(x)u' + q(x)u = f(x)$$
with $h(x) = 1$, $q(x) = 0$, $f(x) = e^x$.

To put it into self adjoint form we create the integrating factor
$$H(x) = \exp\left[\int h(x)\,dx\right]$$
$$= e^x$$
and multiply it through the ODE. giving
$$\frac{d}{dx}\left[e^x \frac{du}{dx}\right] = e^{2x}.$$
We now have $F(x) = e^{2x}$.

To construct the Green function, $G(x,\xi)$ we solve the homogeneous form of the ODE with G replacing u.

1. $G'' + G' = 0 \qquad — (1)$
$$\Rightarrow \frac{d}{dx}\left[e^x \frac{dG}{dx}\right] = 0$$
Integrating both sides with respect to x yields
$$e^x \frac{dG}{dx} = k_1$$
$$\Rightarrow \frac{dG}{dx} = k_1 e^{-x}$$
Integrating both sides with respect to x again, yields
$$G = -k_1 e^{-x} + k_2. \qquad — (2)$$
$$\therefore G(x,\xi) = -k_1(\xi)e^{-x} + k_2(\xi).$$

2. $G(x,\xi)\big|_{x=\xi} = 0$
$$\Rightarrow -k_1(\xi)e^{-\xi} + k_2(\xi) = 0$$

$$\Rightarrow \quad k_2(\xi) = k_1(\xi)e^{-\xi}.$$

$$\therefore G(x,\xi) = k_1(\xi)\left(e^{-\xi} - e^{-x}\right)$$

3. $$\left.\frac{\partial G}{\partial x}(x,\xi)\right|_{x=\xi} = \frac{1}{H(\xi)}$$

$$\Rightarrow \quad k_1(\xi)e^{-\xi} = e^{-\xi}$$

$$\Rightarrow \quad k_1(\xi) = 1$$

$$\therefore G(x,\xi) = \left(e^{-\xi} - e^{-x}\right)$$

Since one of the boundary conditions is nonhomogeneous,

$$u(x) = \int_0^x F(\xi)G(x,\xi)\,d\xi + c_1G_1 + c_2G_2$$

where G_1 and G_2 are the linearly independent solutions (equation 2) of the homogeneous form of the ODE (equation 1).

$$\therefore u(x) = \int_0^x e^{2\xi}\left(e^{-\xi} - e^{-x}\right)d\xi + c_1e^{-x} + c_2$$

$$= \int_0^x \left(e^{\xi} - e^{2\xi - x}\right)d\xi + c_1e^{-x} + c_2$$

$$= \left[e^{\xi} - \frac{e^{2\xi - x}}{2}\right]_0^x + c_1e^{-x} + c_2$$

$$= \left[\left(e^{x} - \frac{e^{x}}{2}\right) - \left(1 - \frac{e^{-x}}{2}\right)\right] + c_1e^{-x} + c_2$$

$$= \frac{e^{x}}{2} + \left(c_1 + \frac{1}{2}\right)e^{-x} + c_2$$

Using the boundary conditions to solve for c_1 and c_2,

$$u'(x) = \frac{e^x}{2} - \left(c_1 + \frac{1}{2}\right)e^{-x}$$

$$u'(0) = 0$$
$$= \frac{1}{2} - \left(c_1 + \frac{1}{2}\right)$$
$$= -c_1$$
$$\therefore c_1 = 0$$

and $$u(x) = \frac{e^x}{2} + \frac{e^{-x}}{2} + c_2$$
$$= \cosh(x) + c_2.$$

$$u(0) = 1$$
$$= \cosh(0) + c_2$$
$$= 1 + c_2$$
$$\Rightarrow c_2 = 0$$

$$\therefore u(x) = \cosh(x).$$

Check:
$$u'(x) = \sinh(x)$$
$$u''(x) = \cosh(x)$$
$$\therefore u''(x) + u'(x) = \cosh(x) + \sinh(x)$$
$$= \frac{e^x}{2} + \frac{e^{-x}}{2} + \frac{e^x}{2} - \frac{e^{-x}}{2}$$
$$= e^x.$$

$\therefore$ the ODE is confirmed.

$$u(0) = \cosh(0) = 1$$
$$u'(0) = \sinh(0) = 0$$

$\therefore$ both boundary conditions are confirmed.

KMA354 Exam 2011

3. $\dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = x \qquad 0<x<1 \quad t>0.$

$$\left.\begin{array}{ll}\text{BC1:} & u(0,t) = 0 \\ \text{BC2:} & u(1,t) = 0\end{array}\right\} t>0.$$

$$\left.\begin{array}{ll}\text{IC1:} & u(x,0) = x \\ \text{IC2:} & \dfrac{\partial u}{\partial t}(x,0) = 0\end{array}\right\} x \in (0,1).$$

Let $u(x,t) = y(x,t) + \psi(x)$

Then $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 y}{\partial t^2}$,

$$\frac{\partial u}{\partial x} = \frac{\partial y}{\partial x} + \frac{d\psi}{dx}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 y}{\partial x^2} + \frac{d^2\psi}{dx^2}.$$

$\therefore$ PDE becomes

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} - \frac{d^2\psi}{dx^2} = x.$$

$$\Rightarrow \frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = \frac{d^2\psi}{dx^2} + x.$$

Let both sides equal zero, then

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = 0 \quad \text{and} \quad \frac{d^2\psi}{dx^2} = -x.$$

BC1: $u(0,t) = 0 = y(0,t) + \psi(0) = 0$

Let $y(0,t) = 0$ and $\psi(0) = 0$.

BC2: $u(1,t) = 0 = y(1,t) + \psi(1) = 0$

Let $y(1,t) = 0$ and $\psi(1) = 0$.

IC1: $u(x,0) = x = y(x,0) + \psi(x)$

$\therefore y(x,0) = x - \psi(x)$

IC2: $\dfrac{\partial u}{\partial t}(x,0) = 0 = \dfrac{\partial y}{\partial t}(x,0)$

Subproblem 1:

$$\frac{d^2\psi}{dx^2} + x = 0$$

BC1: $\psi(0) = 0$

BC2: $\psi(1) = 0.$

$$\psi'' = -x$$
$$\Rightarrow \psi' = -\frac{x^2}{2} + c_1$$
$$\Rightarrow \psi = -\frac{x^3}{6} + c_1 x + c_2$$
$$\psi(0) = 0 = c_2 \qquad \therefore \psi = -\frac{x^3}{6} + c_1 x.$$
$$\psi(1) = 0 = -\frac{1}{6} + c_1 \Rightarrow c_1 = \frac{1}{6}.$$
$$\therefore \psi = -\frac{x^3}{6} + \frac{x}{6} = \frac{x}{6}(1-x^2)$$

Subproblem 2:
$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = 0$$
BC1: $y(0,t) = 0$ IC1: $y(x,0) = x - \psi(x)$
BC2: $y(1,t) = 0$ IC2: $y_t(x,0) = 0.$

Let $y(x,t) = X(x)T(t)$

Then $\frac{\partial^2 y}{\partial t^2} = X\frac{d^2T}{dt^2}$

and $\frac{\partial^2 y}{\partial x^2} = T\frac{d^2X}{dx^2}.$

$\therefore$ PDE becomes $XT'' - TX'' = 0.$
$$\Rightarrow \frac{T''}{T} = \frac{X''}{X}$$
Differentiating both sides with respect to x (or t) must equate to zero. Hence
$$\frac{T''}{T} = \frac{X''}{X} = k, \text{ a separation constant}$$
$$\Rightarrow T'' - kT = 0$$
$$X'' - kX = 0.$$
Based on the form of the boundary / initial conditions, let $k = -\lambda^2 < 0$
$$\therefore T'' + \lambda^2 T = 0 \qquad — ①$$
$$X'' + \lambda^2 X = 0 \qquad — ②$$
BC1: $y(0,t) = 0 = X(0)T(t)$
$T(t) \neq 0 \ \forall t \quad \therefore X(0) = 0.$
BC2: $y(1,t) = 0 = X(1)T(t)$

$T(t) \neq 0 \ \forall t \qquad \therefore X(1) = 0.$

Equation ② has solution

$$X(x) = a_1 \cos(\lambda x) + a_2 \sin(\lambda x).$$

With BC1, $X(0) = 0 = a_1$

$\therefore X(x) = a_2 \sin(\lambda x).$

With BC2, $X(1) = 0 = a_2 \sin(\lambda x).$

$a_2 \neq 0 \quad \therefore \lambda = n\pi \qquad n = 1, 2, 3, \ldots$

$$\therefore X_n(x) = a_{2n} \sin(n\pi x)$$

IC1: $y(x,0) = x - \psi(x) = x - \frac{x}{6} + \frac{x^3}{6} = \frac{x}{6}(5 + x^2).$

$$y(x,0) = X(x)T(0) = \frac{x}{6}(5 + x^2).$$

IC2: $y_t(x,0) = X(x)T'(0) = 0$

$X(x) \neq 0 \ \forall x \quad \therefore T'(0) = 0.$

Equation ① has solution

$$T(t) = a_3 \cos(\lambda t) + a_4 \sin(\lambda t)$$

$$T'(t) = \lambda(-a_3 \sin(\lambda t) + a_4 \cos(\lambda t))$$

With IC2, $T'(0) = 0 = \lambda(a_4)$

$\Rightarrow a_4 = 0 \quad$ since $\lambda = n\pi \neq 0.$

$\therefore T(t) = a_3 \cos(\lambda t)$

and $T_n(t) = a_{3n} \cos(n\pi t).$

Now $y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty} X_n(x) T_n(t)$

$$= \sum_{n=1}^{\infty} a_{2n} a_{3n} \sin(n\pi x)\cos(n\pi t)$$

$$= \sum_{n=1}^{\infty} \alpha_n \sin(n\pi x)\cos(n\pi t)$$

with $\alpha_n = a_{2n} a_{3n}.$

With IC1, $y(x,0) = \sum_{n=1}^{\infty} \alpha_n \sin(n\pi x) = \frac{x}{6}(5 + x^2).$

Then $\alpha_n = 2\int_0^1 \frac{x}{6}(5 + x^2)\sin(n\pi x)\,dx.$

and $u(x,t) = \frac{x}{6}(1 - x^2) + \sum_{n=1}^{\infty} \alpha_n \sin(n\pi x)\cos(n\pi t).$

4(a) $\frac{dy}{dx} - y = x^2.$ — ①

$x = x_0 = 0$ is an ordinary point so we can use the power series method around $x_0 = 0$.

Let $y = \sum_{m=0}^{\infty} a_m x^m$ — ②

$\therefore \frac{dy}{dx} = \sum_{m=1}^{\infty} a_m m x^{m-1}$ — ③

Substituting ② and ③ into ① yields

$$\sum_{m=1}^{\infty} a_m m x^{m-1} - \sum_{m=0}^{\infty} a_m x^m = x^2$$

$$\Rightarrow \sum_{m=0}^{\infty} a_{m+1}(m+1)x^m - \sum_{m=0}^{\infty} a_m x^m = x^2$$

$$\Rightarrow \sum_{m=0}^{\infty} \left(a_{m+1}(m+1) - a_m\right) x^m = x^2.$$

Equating powers of x on both sides of the equality:

$$a_{m+1}(m+1) - a_m = \begin{cases} 0 & m \neq 2 \\ 1 & m = 2. \end{cases}$$

$\therefore$ for $m \neq 2$, $a_{m+1} = \frac{a_m}{m+1}$

for $m = 2$, $a_{m+1} = \frac{a_m + 1}{m+1}$

$m = 0$, $a_1 = \frac{a_0}{1}$

$m = 1$, $a_2 = \frac{a_1}{2} = \frac{a_0}{2 \times 1} = \frac{a_0}{2!}$

$$m=2, \quad a_3 = \frac{a_2+1}{3} = \frac{1}{3}\left(\frac{a_0}{2!}+1\right)$$
$$= \frac{a_0}{3!} + \frac{1}{3}$$
$$= \frac{a_0}{3!} + \frac{2}{3\times 2}$$
$$= \frac{a_0+2}{3!}$$
$$m=3, \quad a_4 = \frac{a_3}{4} = \frac{a_0+2}{4\times 3!} = \frac{a_0+2}{4!}$$
$$m=4, \quad a_5 = \frac{a_4}{5} = \frac{a_0+2}{5\times 4!} = \frac{a_0+2}{5!}$$

$$\therefore \quad a_k = \begin{cases} \dfrac{a_0}{k!} & k = 0,1,2. \\ \dfrac{a_0}{k!} + \dfrac{2}{k!} & k \geqslant 3. \end{cases}$$

Substituting into (2),

$$y = \sum_{m=0}^{\infty} a_m x^m$$
$$= \sum_{m=0}^{2} \frac{a_0}{m!} x^m + \sum_{m=3}^{\infty}\left(\frac{a_0}{m!} + \frac{2}{m!}\right)x^m$$
$$= \sum_{m=0}^{2}\left(\frac{a_0}{m!} + \frac{2}{m!} - \frac{2}{m!}\right)x^m + \sum_{m=3}^{\infty}\left(\frac{a_0}{m!} + \frac{2}{m!}\right)x^m$$
$$= \sum_{m=0}^{\infty}\left(\frac{a_0}{m!} + \frac{2}{m!}\right)x^m - \sum_{m=0}^{2} \frac{2}{m!} x^m$$
$$= (a_0+2)\sum_{m=0}^{\infty} \frac{x^m}{m!} - 2\left(1+x+x^2\right)$$
$$= Ae^x - 2(1+x+x^2)$$

where $A = a_0+2$ and $\sum_{m=0}^{\infty} \frac{x^m}{m!} = e^x$

(b) $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - k^2 y = 0.$

$\Rightarrow \quad \frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{k^2}{x^2}y = 0.$

$x=0$ is a singularity.

1. $\lim_{x\to 0} \frac{x}{x} = \lim_{x\to 0} 1 = 1$ $\qquad \lim_{x\to x_0} \frac{x-x_0}{x}$

2. $\lim_{x\to 0} \frac{x^2}{x^2} = \lim_{x\to 0} 1 = 1.$ $\qquad \lim_{x\to x_0} \frac{(x-x_0)^2}{x^2}$

Since both tests produce finite limits, $x=0$ is a regular singular point.

5(a) $$\frac{d}{dx}\left[H(x)\frac{dy}{dx}\right] + \left(Q(x) + \lambda W(x)\right)y = 0$$

$$-\infty < a \le x \le b < \infty.$$

$$a_1 y(a) + a_2 y'(a) = 0$$
$$b_1 y(b) + b_2 y'(b) = 0.$$

This Sturm Liouville boundary value problem is satisfied by a set of eigenfunctions $y_j(x)$ with corresponding eigenvalues λ_j.

Suppose we take two of these eigenfunctions y_j and y_k and their respective eigenfunctions λ_j and λ_k, both eigenfunction/eigenvalue combination must satisfy the DE; i.e.

$$\frac{d}{dx}\left[H(x)y_j'\right] + \left(Q(x) + \lambda_j W(x)\right)y_j = 0 \quad -①$$
$$\frac{d}{dx}\left[H(x)y_k'\right] + \left(Q(x) + \lambda_k W(x)\right)y_k = 0 \quad -②$$

If we take ① and multiply it by y_k, take ② and multiply it by y_j, subtract the two products and integrate over the interval $x \in [a,b]$, we obtain the expression

$$(\lambda_j - \lambda_k)\int_a^b W(x)\,y_j(x)\,y_k(x)\,dx$$
$$= \left[H(x)\left(y_j(x)y_k'(x) - y_k(x)y_j'(x)\right)\right]_a^b.$$

When $j=k$, $\lambda_j - \lambda_k = 0$ and $y_j y_k' - y_k y_j' = 0$.

Also, $\int_a^b W(x)\,y_j(x)\,y_k(x)\,dx \neq 0$

because $\int_a^b w(x)\, y_j^2(x)\, dx = 0$

implies $w(x) = 0$ or $y_j(x) = 0$; both are situations that can not happen.

When $j \neq k$, $\lambda_j - \lambda_k \neq 0$ so

let $\int_a^b w(x)\, y_j(x)\, y_k(x)\, dx = 0$

$$\Rightarrow \left[H(x) \left(y_j(x)\, y_k'(x) - y_k(x)\, y_j'(x) \right) \right]_a^b = 0.$$

$$\Rightarrow H(b) \left(y_j(b)\, y_k'(b) - y_k(b)\, y_j'(b) \right) - H(a) \left(y_j(a)\, y_k'(a) - y_k(a)\, y_j'(a) \right) = 0$$

There are various ways to enforce this condition, all stemming from the way in which the boundary conditions are specified.

(b) $$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \ell(\ell+1)y = 0 \qquad -①$$

$$\Rightarrow \frac{d^2y}{dx^2} + \left(\frac{-2x}{1-x^2}\right)\frac{dy}{dx} + \frac{\ell(\ell+1)}{(1-x^2)}y = 0 \qquad -②$$

Constructing the integrating factor,

$$\begin{aligned} H(x) &= \exp\left[\int \frac{-2x}{1-x^2}\,dx\right] \\ &= \exp\left[\ln|1-x^2|\right] \\ &= 1-x^2. \end{aligned}$$

In self-adjoint form, equation ② becomes

$$\frac{d}{dx}\Big[\underbrace{(1-x^2)}_{H(x)}\frac{dy}{dx}\Big] + \big(\underbrace{0}_{Q(x)} + \ell(\ell+1)\underbrace{\,}_{w(x)}\big)y = 0.$$

The weight function $w(x) = 1$ $\left(= H(x)\dfrac{1}{1-x^2}\right)$ and the eigenvalues are $\lambda = \ell(\ell+1)$. For integer ℓ, the Legendre polynomials $P_\ell(x)$ are solutions to the equation.

The polynomials are orthogonal on the interval $(-1,1)$ with respect to the weight function $w(x)=1$. Inspection of equation ① shows the equation to be singular at $x=\pm 1$. With the power series constructed about $x_0=0$, orthogonality states

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx \begin{cases} = 0 & m \neq n \\ \neq 0 & m = n \end{cases}$$