Stationary distribution of the tandem fluid queue and its application* to the accumulated priority queue

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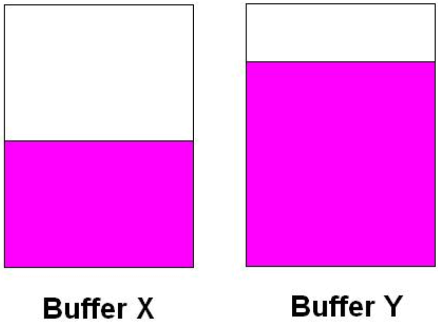
Outline

TFQ Model: two fluid queues driven by $\varphi(t)$

- CTMC $\{\varphi(t) : t \geq 0\}$ with finite state space S, generator $\mathbf{T}$
- Two fluid queues, contents $X(t)$ and $Y(t)$, both $\in [0, \infty)$



First queue $X(t)$ driven by $\varphi(t)$

- $(\varphi(t), X(t))$ is standard fluid queue
- Fluid rates in $\mathbf{R} = diag(r_i)_{i\in\mathcal{S}}$

$$\frac{d}{dt}X(t) = r_{\varphi(t)} \qquad \text{when } X(t) > 0,$$
$$\frac{d}{dt}X(t) = \max(0, r_{\varphi(t)}) \qquad \text{when } X(t) = 0.$$

- $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_- \cup \mathcal{S}_\bigcirc$, e.g. $\mathcal{S}_+ = \{i \in \mathcal{S} : r_i > 0\}$
  (upstates, downstates, zero-states)
- also: $\mathcal{S}_\ominus = \mathcal{S}_- \cup \mathcal{S}_\bigcirc$ ("zero-states at $X(t) = 0$")
- after ordering,

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_{++} & \mathbf{T}_{+-} & \mathbf{T}_{+\bigcirc} \\ \mathbf{T}_{-+} & \mathbf{T}_{--} & \mathbf{T}_{-\bigcirc} \\ \mathbf{T}_{\bigcirc+} & \mathbf{T}_{\bigcirc-} & \mathbf{T}_{\bigcirc\bigcirc} \end{bmatrix}.$$

Second queue $Y(t)$ driven by $(\varphi(t), X(t))$

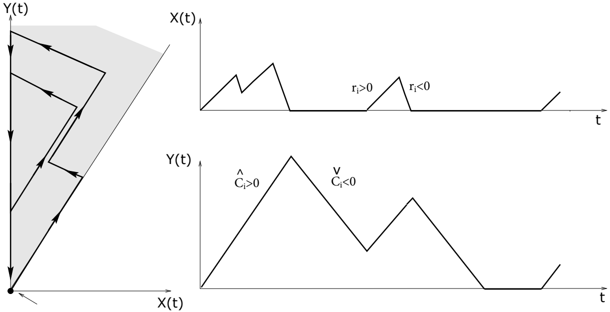
- $\widehat{\mathbf{C}} = diag(\widehat{c}_i)_{i\in\mathcal{S}}$, $\widehat{c}_i > 0$, and
- $\check{\mathbf{C}} = diag(\check{c}_i)_{i\in\mathcal{S}_\ominus}$, $\check{c}_i < 0$.

- $Y(t)$ increases when $X(t) > 0$, at rate $\widehat{c}_{\varphi(t)}$
- $Y(t)$ decreases when $X(t) = 0$, at rate $\check{c}_{\varphi(t)}$ (unless $Y(t) = 0$).

That is,

$$\begin{aligned}
\frac{d}{dt}Y(t) &= \widehat{c}_{\varphi(t)} > 0 && \text{when } X(t) > 0,\\
\frac{d}{dt}Y(t) &= \check{c}_{\varphi(t)} < 0 && \text{when } X(t) = 0,\ Y(t) > 0,\\
\frac{d}{dt}Y(t) &= \widehat{c}_{\varphi(t)} \cdot 1\{\varphi(t) \in \mathcal{S}_+\} && \text{when } X(t) = 0,\ Y(t) = 0.
\end{aligned}$$

Qualitative behaviour

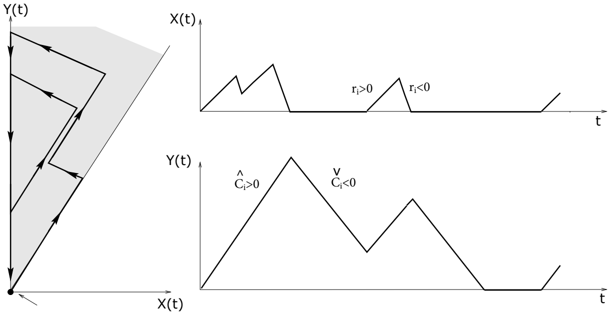


Assuming stability (see paper) process $(\varphi(t), X(t), Y(t))$ alternates between:

(i) periods on $x = 0$

(ii) periods on $x > 0$

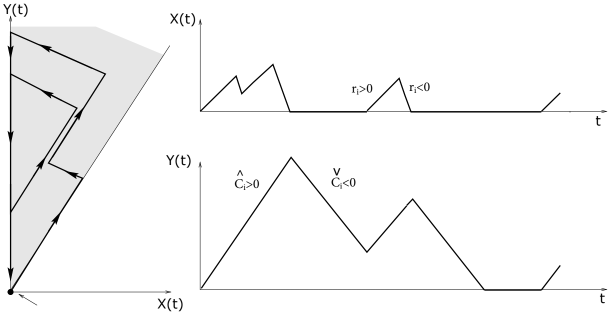
Qualitative behaviour (i) on $x = 0$



(i) periods on $x = 0$

- $Y(t)$ decreasing, unless at $x = 0, y = 0$
- $\varphi(t)$ in $\mathcal{S}_\ominus$
- starts at $x = 0, y > 0$, with $\varphi(t)$ in $\mathcal{S}_-$
- ends at $x = 0, y \geq 0$, with $\varphi(t)$ jumping from $\mathcal{S}_\ominus$ to $\mathcal{S}_+$

Qualitative behaviour (ii) on $x > 0$



(ii) periods on $x > 0$

- $Y(t)$ increasing (while $X(t)$ can either increase or decrease)
- $\varphi(t)$ in $\mathcal{S}$ (any phase)
- starts at $x = 0, y \geq 0$, with $\varphi(t) \in \mathcal{S}_+$
- ends at $x = 0, y > 0$, with $\varphi(t) \in \mathcal{S}_-$

Stationary distribution

has following form (all *vectors* with $|\mathcal{S}|$ components):

(i)
- 1-dimensional densities $\boldsymbol{\pi}(0, y)$
  at $x = 0, y > 0$
- point masses $\mathbf{p}(0, 0)$
  at $(0, 0)$

(ii)
- 2-dimensional densities $\boldsymbol{\pi}(x, y)$
  on $\{(x, y) : x > 0, y > x \cdot \min_{i \in \mathcal{S}_+} \{\widehat{c}_i / r_i\}\}$
- 1-dimensional density $\pi^i(x, x\widehat{c}_i / r_i)$
  on line $y = x\widehat{c}_i / r_i$, $i \in \mathcal{S}_+$

Approach

- Introduce embedded discrete-time process J_k
- Find its stationary distribution ξ_y

- Express $\boldsymbol{\pi}(0, y)$ and $\mathbf{p}(0, 0)$ in ξ_y, using down-shift in Y
- Normalise based on knowledge of $(\varphi(t), X(t))$

- Express $\boldsymbol{\pi}(x, y)$ in $\boldsymbol{\pi}(0, y)$ and $\mathbf{p}(0, 0)$, using up-shift in Y
- Express $\pi^i(x, x\widehat{c}_i/r_i)$ in $\mathbf{p}(0, 0)$

Mostly as LST's (but not always)

(i) down-shift: $\breve{\mathbf{Q}}_{\ominus\ominus}$ and $\breve{\mathbf{Q}}_{\ominus+}$

Let $D(t) = \int_{u=0}^{t} |\breve{c}_{\varphi(u)}| du$ and $t_z = \inf\{t > 0 : D(t) = z\}$.

Define

$$\breve{\mathbf{Q}}_{\ominus\ominus} = (|\breve{\mathbf{C}}_{\ominus}|)^{-1}\mathbf{T}_{\ominus\ominus}, \qquad \breve{\mathbf{Q}}_{\ominus+} = (|\breve{\mathbf{C}}_{\ominus}|)^{-1}\mathbf{T}_{\ominus+}.$$

Then for $i, j \in \mathcal{S}_{\ominus}$, and $z > 0$,

$$[e^{\breve{\mathbf{Q}}_{\ominus\ominus} z}]_{ij} \; = \; P(\varphi(t_z) = j, \varphi(u) \in \mathcal{S}_{\ominus}, 0 \le u \le t_z \mid \varphi(0) = i, X(0) = 0)$$

and $\breve{\mathbf{Q}}_{\ominus+}$ is a matrix of transition rates (w.r.t. level) to phases in $\mathcal{S}_+$ (for times at which X and Y start increasing).

[Bean, O'Reilly and Taylor. Hitting probabilities and hitting times for stochastic fluid flows, *Stochastic Processes and their Applications*, 2005]

(ii) up-shift: $\widehat{\mathbf{Q}}(s)$ and $\widehat{\mathbf{\Psi}}(s)$

Let $\theta = \inf\{t > 0 : X(t) = 0\}$ and $U(t) = \int_{u=0}^{t} \widehat{c}_{\varphi(u)} du$.

Then $U(\theta)$ is total up-shift in Y during Busy Period of X.

Its $|\mathcal{S}_+| \times |\mathcal{S}_-|$ density matrix $\widehat{\psi}(z)$ is given via LST

$$\widehat{\mathbf{\Psi}}(s) = \int_{z=0}^{\infty} e^{-sz} \widehat{\psi}(z) dz$$

with

$$[\widehat{\mathbf{\Psi}}(s)]_{ij} = E(e^{-sU(\theta)} 1\{\varphi(\theta) = j\} \mid \varphi(0) = i, X(0) = 0).$$

[Bean and O'Reilly. A stochastic two-dimensional fluid model, *Stochastic Models*, 2013]

(ii) up-shift: $\widehat{\mathbf{Q}}(s)$ and $\widehat{\boldsymbol{\Psi}}(s)$

To find $\widehat{\boldsymbol{\Psi}}(s)$ define Key generator matrix

$$\widehat{\mathbf{Q}}(s) = \begin{bmatrix} \widehat{\mathbf{Q}}(s)_{++} & \widehat{\mathbf{Q}}(s)_{+-} \\ \widehat{\mathbf{Q}}(s)_{-+} & \widehat{\mathbf{Q}}(s)_{--} \end{bmatrix}$$

$$\widehat{\mathbf{Q}}(s)_{++} = (\mathbf{R}_+)^{-1} \left(\mathbf{T}_{++} - s\widehat{\mathbf{C}}_+ - \mathbf{T}_{+\bigcirc}(\mathbf{T}_{\bigcirc\bigcirc} - s\widehat{\mathbf{C}}_\bigcirc)^{-1}\mathbf{T}_{\bigcirc+} \right)$$
$$\widehat{\mathbf{Q}}(s)_{+-} = (\mathbf{R}_+)^{-1} \left(\mathbf{T}_{+-} - \mathbf{T}_{+\bigcirc}(\mathbf{T}_{\bigcirc\bigcirc} - s\widehat{\mathbf{C}}_\bigcirc)^{-1}\mathbf{T}_{\bigcirc-} \right)$$
$$\widehat{\mathbf{Q}}(s)_{-+} = (|\mathbf{R}_-|)^{-1} \left(\mathbf{T}_{-+} - \mathbf{T}_{-\bigcirc}(\mathbf{T}_{\bigcirc\bigcirc} - s\widehat{\mathbf{C}}_\bigcirc)^{-1}\mathbf{T}_{\bigcirc+} \right)$$
$$\widehat{\mathbf{Q}}(s)_{--} = (|\mathbf{R}_-|)^{-1} \left(\mathbf{T}_{--} - s\widehat{\mathbf{C}}_- - \mathbf{T}_{-\bigcirc}(\mathbf{T}_{\bigcirc\bigcirc} - s\widehat{\mathbf{C}}_\bigcirc)^{-1}\mathbf{T}_{\bigcirc-} \right).$$

Then $\widehat{\boldsymbol{\Psi}}(s)$ is minimum nonnegative solution of Riccati eq.

$$\widehat{\mathbf{Q}}(s)_{+-} + \widehat{\mathbf{Q}}(s)_{++}\widehat{\boldsymbol{\Psi}}(s) + \widehat{\boldsymbol{\Psi}}(s)\widehat{\mathbf{Q}}(s)_{--} + \widehat{\boldsymbol{\Psi}}(s)\widehat{\mathbf{Q}}(s)_{-+}\widehat{\boldsymbol{\Psi}}(s) = \mathbf{O}.$$

[Bean and O'Reilly. A stochastic two-dimensional fluid model, *Stochastic Models*, 2013]

Embedded process J_k

Let $J_k = (\varphi(\theta_k), Y(\theta_k))$ be with state space $\mathcal{S}_- \times (0, \infty)$, where θ_k is k-th time that $(\varphi(t), X(t), Y(t))$ hits $x = 0$.

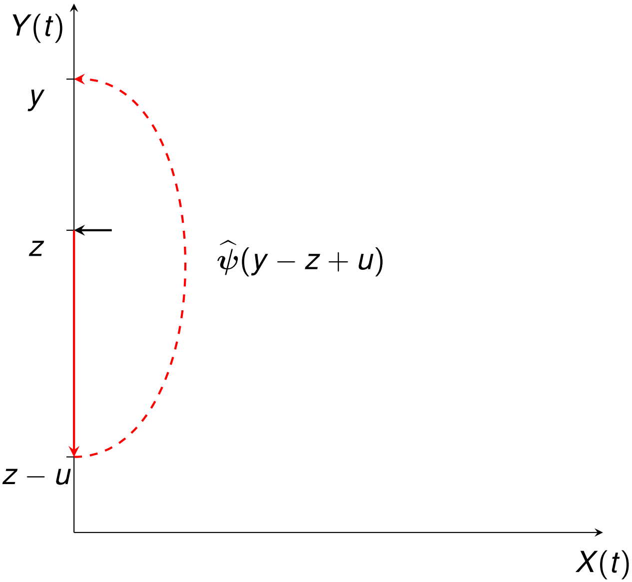
Lemma

The transition kernel of J_k is given by

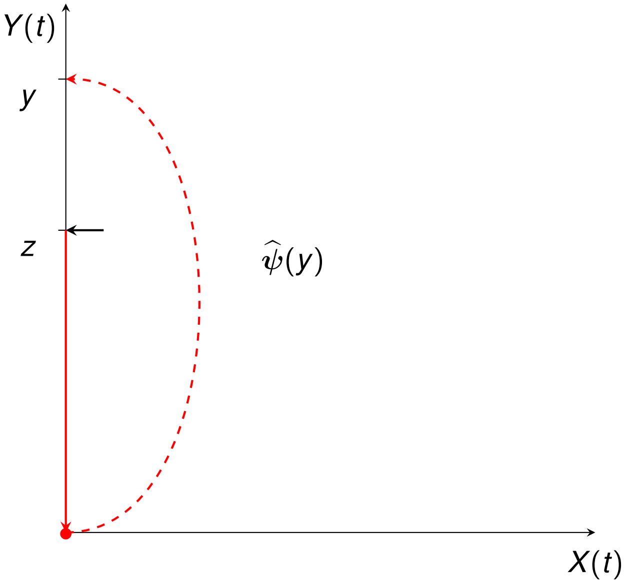
$$\begin{aligned}\mathbf{P}_{z,y} &= \int_{u=[z-y]^+}^{z} \left[\begin{array}{cc}\mathbf{I} & \mathbf{O}\end{array}\right] e^{\check{\mathbf{Q}}_{\ominus\ominus} u} \check{\mathbf{Q}}_{\ominus +} \widehat{\psi}(y - z + u) du \\ &\quad + \left[\begin{array}{cc}\mathbf{I} & \mathbf{O}\end{array}\right] e^{\check{\mathbf{Q}}_{\ominus\ominus} z} (-\check{\mathbf{Q}}_{\ominus\ominus})^{-1} \check{\mathbf{Q}}_{\ominus +} \widehat{\psi}(y)\end{aligned}$$

where $[x]^+$ *denotes* $\max(0, x)$*, and* $\left[\begin{array}{cc}\mathbf{I} & \mathbf{O}\end{array}\right]$ *is a* $|\mathcal{S}_-| \times |\mathcal{S}_\ominus|$ *matrix.*

Embedded process J_k



Embedded process J_k



Embedded process J_k

Corollary

The Laplace-Stieltjes transform of $\mathbf{P}_{z,y}$ *w.r.t.* y *is given by*

$$\begin{aligned}
\mathbf{P}_{z,\cdot}(s) &= \left[\begin{array}{cc} \mathbf{I} & \mathbf{O} \end{array}\right] e^{-sz}\left(\check{\mathbf{Q}}_{\ominus\ominus} + s\mathbf{I}\right)^{-1}\left(e^{\left(\check{\mathbf{Q}}_{\ominus\ominus}+s\mathbf{I}\right)z} - \mathbf{I}\right) \\
&\quad \times \check{\mathbf{Q}}_{\ominus+}\widehat{\boldsymbol{\Psi}}(s) \\
&\quad + \left[\begin{array}{cc} \mathbf{I} & \mathbf{O} \end{array}\right] e^{\check{\mathbf{Q}}_{\ominus\ominus}z}(-\check{\mathbf{Q}}_{\ominus\ominus})^{-1}\check{\mathbf{Q}}_{\ominus+}\widehat{\boldsymbol{\Psi}}(s).
\end{aligned}$$

Embedded process J_k

Stationary distribution of J_k is given by row vector $\boldsymbol{\xi}_z = [\xi_{i,z}]_{i \in \mathcal{S}_-}$ of densities, satisfying

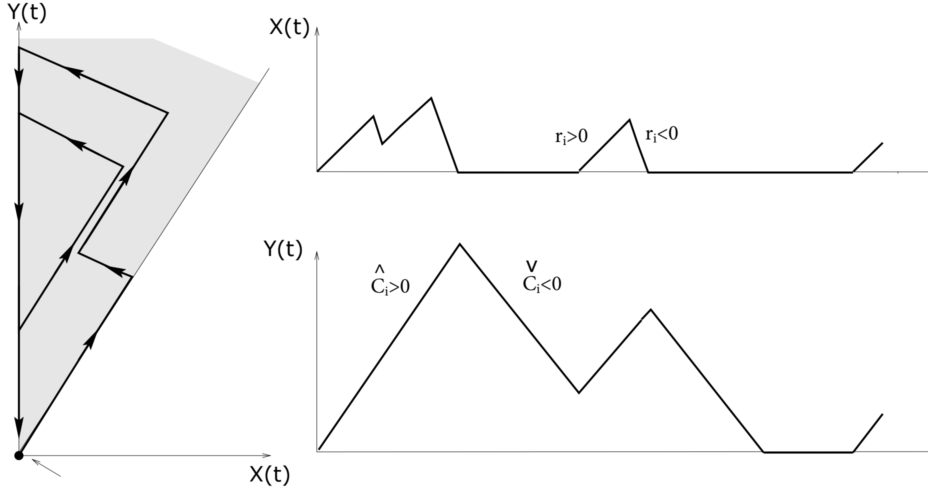
$$\begin{cases} \int_{z=0}^{\infty} \boldsymbol{\xi}_z \mathbf{P}_{z,y} dz & = & \boldsymbol{\xi}_y \\ \int_{y=0}^{\infty} \boldsymbol{\xi}_y dy \mathbf{1} & = & 1 \end{cases}$$

will be solved numerically.

Next step:

Express stationary distribution of $(\varphi(t), X(t), Y(t))$ at level $x = 0$ in terms of $\boldsymbol{\xi}_z$.

Expressing $\pi(0, y)$ and $\mathbf{p}(0, 0)$ in ξ_y



Expressing $\pi(0, y)$ and $\mathbf{p}(0, 0)$ in ξ_y

Lemma

We have $\pi(0, y) = \begin{bmatrix} \mathbf{0} & \pi(0, y)_\ominus \end{bmatrix}$, *where*

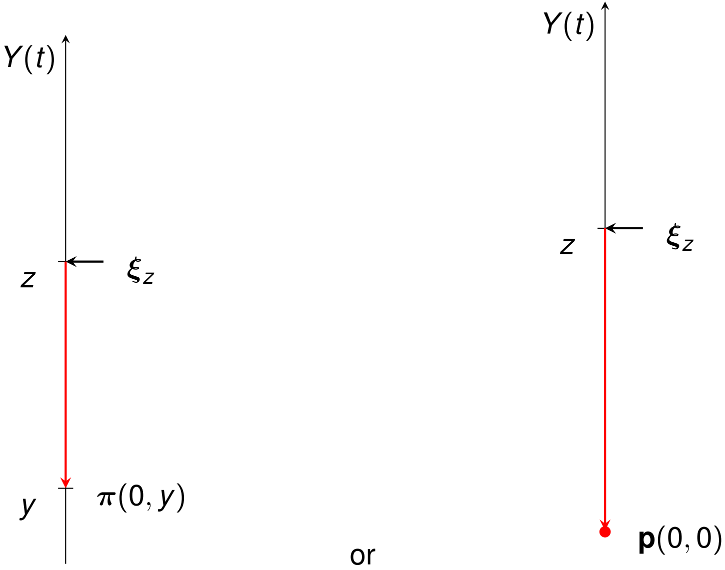
$$\pi(0, y)_\ominus = \alpha \int_{z=y}^{\infty} \begin{bmatrix} \xi_z & \mathbf{0} \end{bmatrix} e^{\check{\mathbf{Q}}_{\ominus\ominus}(z-y)} (|\check{\mathbf{C}}_\ominus|)^{-1} dz,$$

and $\mathbf{p}(0, 0) = \begin{bmatrix} \mathbf{0} & \mathbf{p}(0, 0)_\ominus \end{bmatrix}$, *where*

$$\mathbf{p}(0, 0)_\ominus = \alpha \int_{z=0}^{\infty} \begin{bmatrix} \xi_z & \mathbf{0} \end{bmatrix} e^{\check{\mathbf{Q}}_{\ominus\ominus} z} dz (-\mathbf{T}_{\ominus\ominus})^{-1}.$$

Here, α is a normalizing constant and the total rate of hitting $x = 0$.

Expressing $\pi(0, y)$ and $\mathbf{p}(0, 0)$ in ξ_y



Expressing $\pi(0, y)$ and $\mathbf{p}(0, 0)$ in ξ_y

Define LST of density part

$$\pi(0, \cdot)(s) = \int_{z=0}^{\infty} e^{-sy} \pi(0, y) dy.$$

Corollary

We have $\pi(0, \cdot)(s) = \left[\begin{array}{cc} \mathbf{0} & \pi(0, \cdot)(s)_{\ominus} \end{array} \right]$, *where*

$$\begin{aligned} \pi(0, \cdot)(s)_{\ominus} &= \alpha \int_{z=0}^{\infty} \left[\begin{array}{cc} \xi_z & \mathbf{0} \end{array} \right] e^{\check{\mathbf{Q}}_{\ominus\ominus} z} (\check{\mathbf{Q}}_{\ominus\ominus} + s\mathbf{I})^{-1} \\ & \quad \times \left(\mathbf{I} - e^{-(\check{\mathbf{Q}}_{\ominus\ominus} + s\mathbf{I}) z} \right) (|\check{\mathbf{C}}_{\ominus}|)^{-1} dz. \end{aligned}$$

Normalise, based on 1-dim fluid queue $(\varphi(t), X(t))$

Lemma

The normalising constant α *is given by*

$$\alpha = \left\{ \begin{bmatrix} \boldsymbol{\xi} & \mathbf{0} \end{bmatrix} (-\mathbf{T}_{\ominus\ominus})^{-1} \left(\mathbf{1} + \mathbf{T}_{\ominus+}\mathbf{K}^{-1} \begin{bmatrix} (\mathbf{R}_{+})^{-1} & \boldsymbol{\Psi}(|\mathbf{R}_{-}|)^{-1} \end{bmatrix} \times \left(\mathbf{1} + \mathbf{T}_{\pm\bigcirc}(-\mathbf{T}_{\bigcirc\bigcirc})^{-1}\mathbf{1} \right) \right) \right\}^{-1},$$

where, $\boldsymbol{\xi} = \int_{z=0}^{\infty} \boldsymbol{\xi}_z dz$, $\boldsymbol{\Psi} = \widehat{\boldsymbol{\Psi}}(s)|_{s=0}$ *and* $\mathbf{K} = \widehat{\mathbf{K}}(s)|_{s=0}$ *with*

$$\widehat{\mathbf{K}}(s) = \widehat{\mathbf{Q}}(s)_{++} + \widehat{\boldsymbol{\Psi}}(s)\widehat{\mathbf{Q}}(s)_{-+}.$$

Normalise, based on 1-dim fluid queue $(\varphi(t), X(t))$

Proof. Integrating $\pi(0, y)$ and adding $\mathbf{p}(0, 0)$ yields the probability mass vector of $\varphi(t)$ at $x = 0$,

$$\begin{bmatrix} \mathbf{p}_- & \mathbf{p}_\bigcirc \end{bmatrix} = \alpha \begin{bmatrix} \boldsymbol{\xi} & \mathbf{0} \end{bmatrix} (-\mathbf{T}_{\ominus\ominus})^{-1} .$$

Similarly, we have expression for density $\pi(x)$ at $x > 0$,

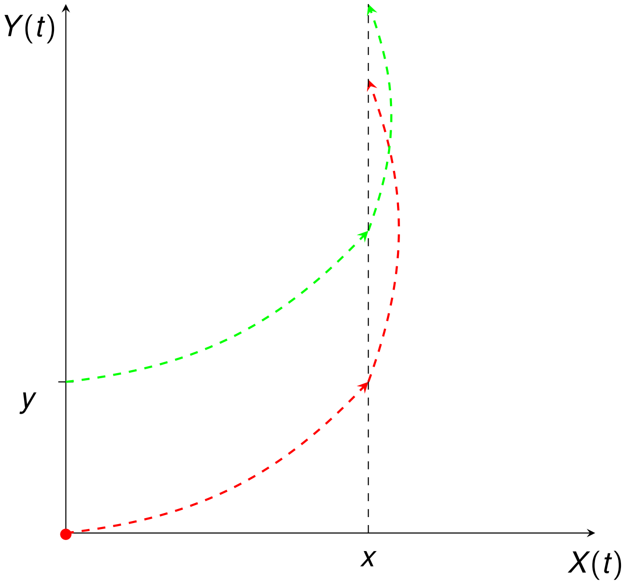
$$\begin{aligned} \begin{bmatrix} \pi(x)_+ & \pi(x)_- \end{bmatrix} &= \begin{bmatrix} \mathbf{p}_- & \mathbf{p}_\bigcirc \end{bmatrix} \mathbf{T}_{\ominus+} e^{\mathbf{K}x} \begin{bmatrix} (\mathbf{R}_+)^{-1} & \boldsymbol{\Psi}(|\mathbf{R}_-|)^{-1} \end{bmatrix} , \\ \pi(x)_\bigcirc &= \begin{bmatrix} \pi(x)_+ & \pi(x)_- \end{bmatrix} \mathbf{T}_{\pm\bigcirc}(-\mathbf{T}_{\bigcirc\bigcirc})^{-1}. \end{aligned}$$

Now solve α from

$$\mathbf{p1} + \int_{x=0}^{\infty} \pi(x) dx \mathbf{1} = 1.$$

□

Expressing $\pi(x, y)$ in $\pi(0, y)$ and $\mathbf{p}(0, 0)$



Expressing $\pi(x, y)$ in $\pi(0, y)$ and $\mathbf{p}(0, 0)$

Lemma

We have

$$\pi(x,\cdot)(s) = \left[\ \pi(x,\cdot)(s)_+ \quad \pi(x,\cdot)(s)_- \quad \pi(x,\cdot)(s)_\bigcirc\ \right]$$

with

$$\begin{aligned}\left[\ \pi(x,\cdot)(s)_+ \quad \pi(x,\cdot)(s)_-\ \right] &= (\pi(0,\cdot)(s)_\ominus + \mathbf{p}(0,0)_\ominus) \\ &\quad \times \mathbf{T}_{\ominus +} e^{\widehat{\mathbf{K}}(s)x} \times \left[\ (\mathbf{R}_+)^{-1} \quad \widehat{\mathbf{\Psi}}(s)(|\mathbf{R}_-|)^{-1}\ \right],\end{aligned}$$

and

$$\begin{aligned}\pi(x,\cdot)(s)_\bigcirc &= \left[\ \pi(x,\cdot)(s)_+ \quad \pi(x,\cdot)(s)_-\ \right] \\ &\quad \times \mathbf{T}_{\pm\bigcirc}(s\widehat{\mathbf{C}}_\bigcirc - \mathbf{T}_{\bigcirc\bigcirc})^{-1}.\end{aligned}$$

Expressing $\pi(x, y)$ in $\pi(0, y)$ and $\mathbf{p}(0, 0)$

Let $\pi(\cdot,\cdot)(v,s) = \int_{x=0}^{\infty} e^{-vx}\pi(x,\cdot)(s)dx$.

Corollary

We have

$$\pi(\cdot,\cdot)(v,s) = \left[\begin{array}{ccc} \pi(\cdot,\cdot)(v,s)_+ & \pi(\cdot,\cdot)(v,s)_- & \pi(\cdot,\cdot)(s)_\bigcirc \end{array}\right]$$

with

$$\left[\begin{array}{cc} \pi(\cdot,\cdot)(v,s)_+ & \pi(\cdot,\cdot)(v,s)_- \end{array}\right] = (\pi(0,\cdot)(s)_\ominus + \mathbf{p}(0,0)_\ominus) \times \left[\begin{array}{c} \mathbf{T}_{-+} \\ \mathbf{T}_{\bigcirc +} \end{array}\right] (-\widehat{\mathbf{K}}(s) + v\mathbf{I})^{-1} \left[\begin{array}{cc} (\mathbf{R}_+)^{-1} & \widehat{\boldsymbol{\Psi}}(s)(|\mathbf{R}_-|)^{-1} \end{array}\right]$$

and

$$\pi(\cdot,\cdot)(s)_\bigcirc = \left[\begin{array}{cc} \pi(\cdot,\cdot)(s)_+ & \pi(\cdot,\cdot)(s)_- \end{array}\right] \mathbf{T}_{\pm\bigcirc} \times (s\widehat{\mathbf{C}}_\bigcirc - \mathbf{T}_{\bigcirc\bigcirc})^{-1}.$$

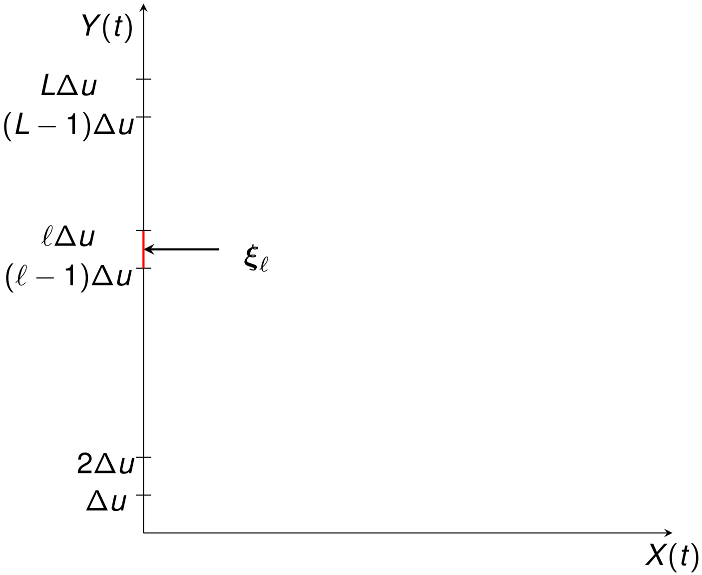
Expressing $\pi^i(x, x\widehat{c}_i/r_i)$ in $\mathbf{p}(0,0)$

Lemma

For all $i \in \mathcal{S}_+$,

$$\pi^i(x, x\widehat{c}_i/r_i) = \sum_{j \in \mathcal{S}_\ominus} \mathbf{p}_j(0,0)\, T_{ji} \exp(-(T_{ii}/r_i)x)/r_i.$$

Numerical scheme: $(\ell - 1)\Delta u \le z \le \ell \Delta u$



Numerical scheme

- Truncate and discretize the state space of J_k to get the DTMC $\{\bar{J}_k : k = 0, 1, 2, \ldots\}$ with state space $\{(i, \ell) : i \in \mathcal{S}_-, \ell = 1, 2, \ldots L\}$

  and matrix $\bar{\mathbf{P}} = [\bar{\mathbf{P}}_{\ell m}]_{\ell, m=0,1,2,\ldots,L}$ made up of block matrices $\bar{\mathbf{P}}_{\ell m} = [\bar{P}_{i,\ell;j,m}]_{i,j\in\mathcal{S}_-}$, where

$$\bar{P}_{i,\ell;j,m} = P(\bar{J}_{k+1} = (j, m) \mid \bar{J}_k = (i, \ell)).$$

- Get $\bar{\mathbf{P}}$ by approximating

$$\bar{\mathbf{P}}_{\ell m} = \int_{y=(m-1)\Delta u}^{m\Delta u} \mathbf{P}_{\ell\Delta u, y} dy \approx \Delta u \mathbf{P}_{\ell\Delta u, m\Delta u},$$

  and normalizing so that $\sum_{m=0}^{L} \bar{\mathbf{P}}_{\ell m}\mathbf{1} = \mathbf{1}$.

- Find $\bar{\boldsymbol{\xi}}_\ell = [\bar{\xi}_{j;\ell}]_{j\in\mathcal{S}_-}$ by solving $\bar{\boldsymbol{\xi}}\bar{\mathbf{P}} = \bar{\boldsymbol{\xi}}, \quad \bar{\boldsymbol{\xi}}\mathbf{1} = \mathbf{1}$.

Numerical scheme

- Use this this to approximate

$$\mathbf{p}(0,0)_{\ominus} \;\approx\; \alpha \sum_{\ell=1}^{L} \left[\; \bar{\xi}_{\ell} \quad \mathbf{0} \;\right] e^{\check{\mathbf{Q}}_{\ominus\ominus}\ell\Delta u}(-\mathbf{T}_{\ominus\ominus})^{-1}$$

and

$$\begin{aligned}\pi(0,\cdot)(s)_{\ominus} \;\approx\; & \alpha \sum_{\ell=1}^{L} \left[\; \bar{\xi}_{\ell} \quad \mathbf{0} \;\right] e^{\check{\mathbf{Q}}_{\ominus\ominus}\ell\Delta u}(\check{\mathbf{Q}}_{\ominus\ominus} + s\mathbf{I})^{-1} \\ & \times \left(\mathbf{I} - e^{-(\check{\mathbf{Q}}_{\ominus\ominus}+s\mathbf{I})\ell\Delta u}\right) (|\check{\mathbf{C}}_{\ominus}|)^{-1}.\end{aligned}$$

- Evaluate $\pi(x,\cdot)(s)$ and invert using Abate and Whitt.

Numerical Example

We consider a process with the following parameters: $\mathcal{S} = \{1, 2\}$, $r_1 = 2$, $r_2 = -6$, $\widehat{c}_1 = \widehat{c}_2 = 2$, $\check{c}_1 = \check{c}_2 = -3$, and

$$\mathbf{T} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}.$$

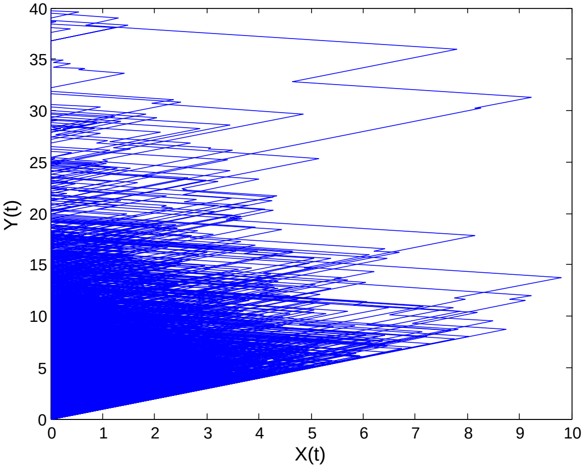
This simple process is similar to the model studied in Kroese and Scheinhardt (2011) and Werner (1998)

(but different from the numerical example analysed there).

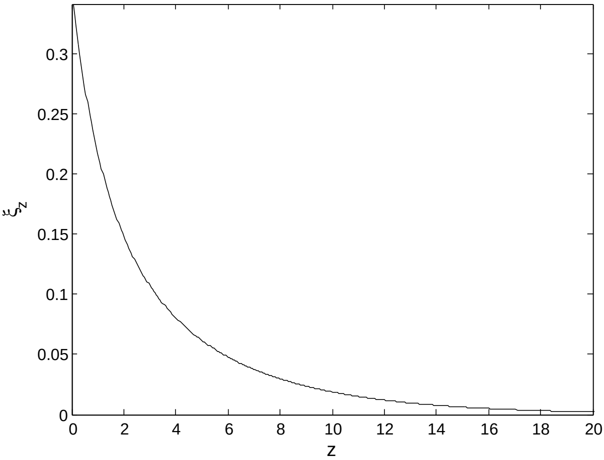
[D.P. Kroese and W.R.W. Scheinhardt. Joint Distributions for Interacting Fluid Queues. *Queueing Systems*, 2001.]

[W.R.W. Scheinhardt, PhD Thesis, 1998.]

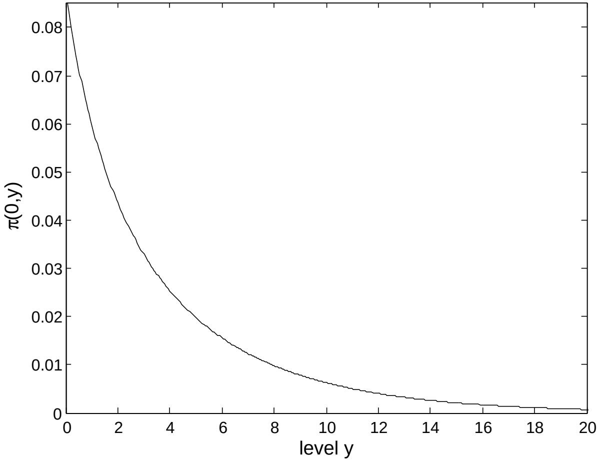
Simulated values $(X(t), Y(t))$, $0 \leq t \leq 10^5$



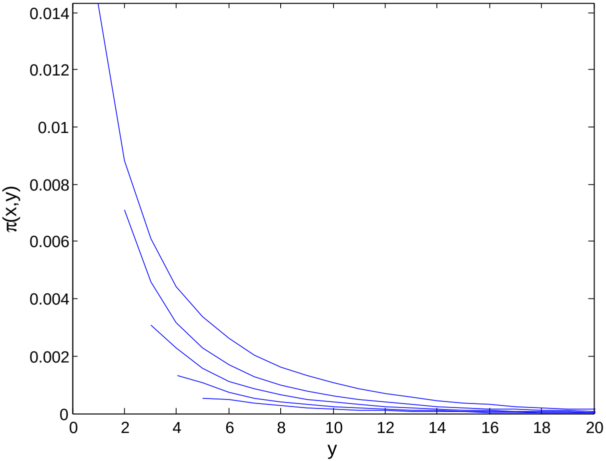
The estimated values $[\xi_z]_j$ for $j = 2$



The estimated values $[\pi(0, y)]_j$ for $j = 2$



The estimated values $[\pi(x,y)]_j$ for $j = 1$, $x = 1, \ldots, 5$



Two-class Accumulating Priority Queue

- Single-server queue with PH service time
- Customer classes $i = 1, 2$
- Poisson arrivals with rates λ_i for $i = 1, 2$
- Class i customer accumulates priority at rate b_i
  (Upon arrival) with $b_1 > b_2$
- Customer with the highest accumulated priority commences service (After completion of service).

Maximum Priority Process **M**

- Let $M_i(t)$ be the least upper bound for all class i customers present in the queue at time t.
- Maximum Priority Process $\mathbf{M} = \{(M_1(t), M_2(t)); t \geq 0\}$.
- We are interested in the stationary distribution of **M** embedded at the moments of commencement of service.

Result

- We map **M** to a certain TFQ $\{(\varphi(t), \widetilde{Z}(t), \widetilde{M}_2(t)); t \geq 0\}$.

- The stationary distribution of $\{(\varphi(t), \widetilde{Z}(t), \widetilde{M}_2(t)); t \geq 0\}$ can be obtained using our results discussed above.

- The stationary distribution of **M** embedded at the moments of commencement of service

  is equivalent to the part of the stationary distribution of
  $\{(\varphi(t), \widetilde{Z}(t), \widetilde{M}_2(t)); t \geq 0\}$
  corresponding to down-phase $-$.

[M. M. O'Reilly and W. R. W. Scheinhardt. Stationary distributions for a class of Markov-modulated tandem fluid queues. *Submitted to Stochastic Models*, 2016.]

Future work

- Numerical analysis for the accumulating priority queue.
- Analysis of the dual model.

1. J. Abate and W. Whitt. Numerical inversion of Laplace transforms of probability distributions. *ORSA Journal of Computing*, 7(1):36–43, 1995.
2. N.G. Bean and M.M. O'Reilly. A stochastic two-dimensional fluid model. *Stochastic Models*, 29(1):31–63, 2013.
3. N.G. Bean and M.M. O'Reilly. The Stochastic Fluid-Fluid Model: A Stochastic Fluid Model driven by an uncountable-state process, which is a Stochastic Fluid Model itself. *Stochastic Processes and Their Applications*, 124(5):1741–1772, 2014.
4. D.P. Kroese and W.R.W. Scheinhardt. Joint Distributions for Interacting Fluid Queues. *Queueing Systems*, 37:99–139, 2001.
5. M. M. O'Reilly and W. R. W. Scheinhardt. Stationary distributions for a class of Markov-modulated tandem fluid queues. *Submitted to Stochastic Models*, 2016.
6. D. A. Stanford, P. Taylor, and I. Ziedins. Waiting time distributions in the accumulating priority queue. *Queueing Systems*, vol. 7, no. 3, pp. 297-330, 2014.
7. W.R.W. Scheinhardt, "Markov-modulated and feedback fluid queues," PhD Thesis, University of Twente, Netherlands, 1998.

Thank you for listening!