Fluid Models

Matrix-Analytic methods in Stochastic Modelling 2004

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Outline

- Model.
- Research so far.
- Future directions.
- References.

From QBDs to fluid models

QBD components

- (N, i), N- level, i- phase,
- Generator Q (special structure, A_0, A_1, A_2)

Note: The level variable is *countable*.

The goal:

A model in which the level variable is *continuous*.

Motivation

Two main reasons:

- Modelling of high-speed communication networks.
- Data in a high-speed communication network buffer behaves like fluid.

A Markov stochastic fluid model

We consider the following *level-independent* Markov process $\{(X(t), \varphi(t)) : t \in \mathcal{R}^+\}$:

- The level is denoted by $X(t) \in \mathcal{R}^+$,
- The phase is denoted by $\varphi(t) \in \mathcal{S}$, $|\mathcal{S}| = m$,
- The phase process $\{\varphi(t) : t \in \mathcal{R}^+\}$ is a Markov chain with infinitesimal generator $\mathcal{T}$.

Net input rates

$$c_i = \frac{dX(t)}{dt}\Big|_{t=0}$$

The rate c_i at which the level of the fluid increases, or decreases, is governed by the state $i \in \mathcal{S}$ of the underlying continous-time Markov chain.

The parameters c_i can be positive, negative or zero.

Two models

General: $c_i \in \mathcal{R}$.

Let

$$\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_0,$$

where

$$\begin{aligned} \mathcal{S}_1 &= \{i : c_i > 0\}, \\ \mathcal{S}_2 &= \{i : c_i < 0\}, \\ \mathcal{S}_0 &= \{i : c_i = 0\}. \end{aligned}$$

Simplified: $c_i = \pm 1$, $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$.

General model $\longrightarrow$ simplified model

(Simplified model is much easier to analyse.)

- A mapping from a general to a model with non-zero rates (Asmussen 1995).
- A model with non-zero rates can be easily transformed into a simplified model (Rogers 1994).

This transformation preserves probabilities but not times!

Asmussen (1994)

- $\mathcal{S}_{old} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_0$, $\quad c_i \in \mathcal{R}$, $\quad i \in \mathcal{S}$,

$$\mathcal{T}_{old} = \begin{bmatrix} T_{00} & T_{01} & T_{02} \\ T_{10} & T_{11} & T_{12} \\ T_{20} & T_{21} & T_{22} \end{bmatrix}$$

- $\mathcal{S}_{new} = \mathcal{S}_1 \cup \mathcal{S}_2$, $\quad c_i \in \mathcal{R} \setminus \{0\}$, $\quad i \in \mathcal{S}$,

$$\mathcal{T}_{new} = \begin{bmatrix} T_{11} - T_{10}T_{00}^{-1}T_{01} & T_{12} - T_{10}T_{00}^{-1}T_{02} \\ T_{21} - T_{20}T_{00}^{-1}T_{01} & T_{22} - T_{20}T_{00}^{-1}T_{02} \end{bmatrix}$$

Rogers (1994)

- $c_i \in \mathcal{R} \setminus \{0\}, \quad i \in \mathcal{S}$,

$$\mathcal{T}_{old} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

- $c_i = \pm 1, \quad i \in \mathcal{S}$,

$$\mathcal{T}_{new} = A\mathcal{T}_{old},$$

where $A = diag(\frac{1}{|c_i|} : i \in \mathcal{S})$.

Example 1

$$\mathcal{T} = \left[\begin{array}{r|r} -2 & 2 \\ \hline 1 & -1 \end{array}\right]$$

$\mathcal{S}_1 = \{1\}$, $c_1 = 1$

$\mathcal{S}_2 = \{2\}$, $c_2 = -1$.

Notation: partitioning of generator $\mathcal{T}$

$$\mathcal{T} = \left[\begin{array}{c|c} T_{11} & T_{12} \\ \hline T_{21} & T_{22} \end{array}\right]$$

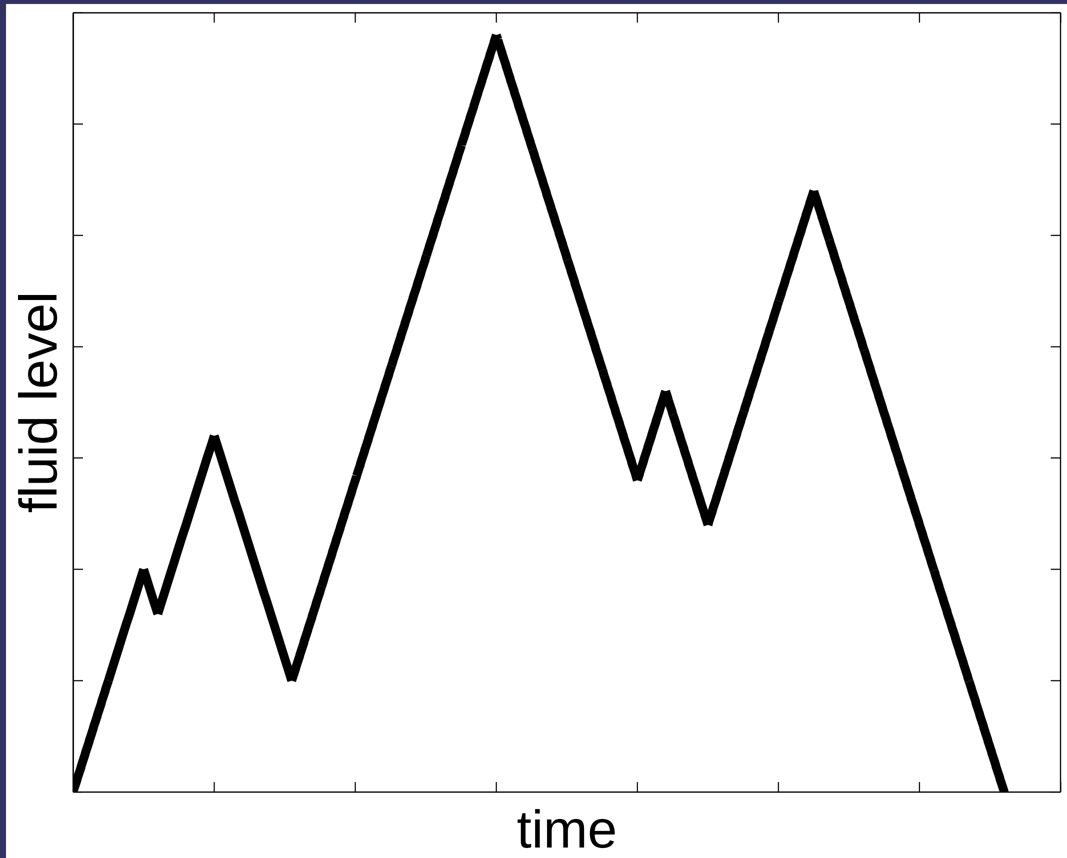
Example 2

$$\mathcal{T} = \left[\begin{array}{cc|ccc} -28 & 22 & 2 & 2 & 2 \\ 21 & -27 & 2 & 2 & 2 \\ \hline 1 & 1 & -26 & 22 & 2 \\ 1 & 1 & 21 & -24 & 1 \\ 1 & 1 & 21 & 1 & -24 \end{array}\right]$$

$\mathcal{S}_1 = \{1, 2\}$, $c_1 = c_2 = 1$

$\mathcal{S}_2 = \{3, 4, 5\}$, $c_3 = c_4 = c_5 = -1$.

Return to the initial level zero



Very useful property:

The model is upward-homogenous!

Important matrix

For any level z, let $\theta(z)$ denote the time in $(0,\infty)$ at which the process first hits level z.

For all $i \in \mathcal{S}_1$, $j \in \mathcal{S}_2$, we define

$$[\Psi]_{ij} = P[\varphi(\theta(0)) = j | X(0) = 0, \varphi(0) = i].$$

Ψ records the probabilities of return journey to the initial level.

Significance:

Ψ appears in the formulae for many performance measures!

Drift - a physical concept

Assuming $+1/-1$ rates, let

$$(1) \quad \mathcal{T} = \left[\begin{array}{r|r} -2 & 2 \\ \hline 1 & -1 \end{array}\right],$$

$$(2) \quad \mathcal{T} = \left[\begin{array}{r|r} -1 & 1 \\ \hline 2 & -2 \end{array}\right],$$

$$(3) \quad \mathcal{T} = \left[\begin{array}{r|r} -1 & 1 \\ \hline 1 & -1 \end{array}\right].$$

Recurrence measure μ

(Simplified model)

$$\mu = \widetilde{\nu_1}\widetilde{e} - \widetilde{\nu_2}\widetilde{e}$$

$(\widetilde{\nu_1}, \widetilde{\nu_2})$ - the stationary distribution vector of the process $\varphi(t)$

(satisfying the equation $(\widetilde{\nu_1}, \widetilde{\nu_2})[\mathcal{T} : \widetilde{e}] = [0 : 1]$),

$\widetilde{e}$ - the column vector of ones.

1. Downward drift $\equiv$ positive recurrent $\equiv \mu < 0$,
2. Upward drift $\equiv$ transient $\equiv \mu > 0$,
3. No drift $\equiv$ null-recurrent $\equiv \mu = 0$.

Bean, O'Reilly and Taylor

Laplace-Stieltjes transforms for several time-related performance measures (general model):

- Times of return journey to the initial level.
- Times of draining/filling to a given level.
- Times of a journey to a given level while avoiding the upper/lower taboo level.
- Expected sojourn times in specified sets.

Steady state densities

For all $j \in \mathcal{S}$, $x > 0$, steady state densities are defined as

$$\pi_j(x) = \lim_{t \to \infty} f_j(t, x),$$

where

$$f_j(t, x) = P[x < X(t) < x + dx, \varphi(t) = j].$$

Notation

Matrix notation is introduced to simplify the analysis:

$$\widetilde{\pi(x)} = (\pi_1(x), \ldots \pi_m(x)), \quad \text{where } |\mathcal{S}| = m,$$

$$C = diag(c_i : i \in \mathcal{S}).$$

Ramaswami (1999)

- From partial differential equations Ramaswami derived the differential equation

$$\widetilde{\pi(x)}\mathcal{T} = \frac{d}{dx}\widetilde{\pi(x)}C$$

  This equation is difficult to solve.

- Ramaswami considered appropriate taboo processes and derived an explicit formula for $\widetilde{\pi(x)}$.

Ramaswami's conditioning.

- Assume that the process starts in $(0, i)$.
- Note that the fluid can reach $x + y$ only after it has crossed x.
- Let $[\phi(\tau, x, x + y)]_{ij}$ be the density of being at $(x + y, j)$ at time τ avoiding the set $[0, x] \times \{1, \ldots, m\}$ in the interval $(0, \tau)$.
- By conditioning on the last epoch of crossing the level x,

$$f_j(t, x + y) = \int_0^t \sum_{i \in \mathcal{S}} f_i(t - \tau, x)[\phi(\tau, x, x + y)]_{ij} d\tau.$$

For more details of the method see Ramaswami (1999).

Expression for $\widetilde{\pi(x)}$ (Ramaswami 1999)

$$(\widetilde{\pi_1(x)}, \widetilde{\pi_2(x)}) = -\widetilde{\nu_1}(T_{11} + \Psi T_{21})[e^{(T_{11}+\Psi T_{21})x}, e^{(T_{11}+\Psi T_{21})x}\Psi].$$

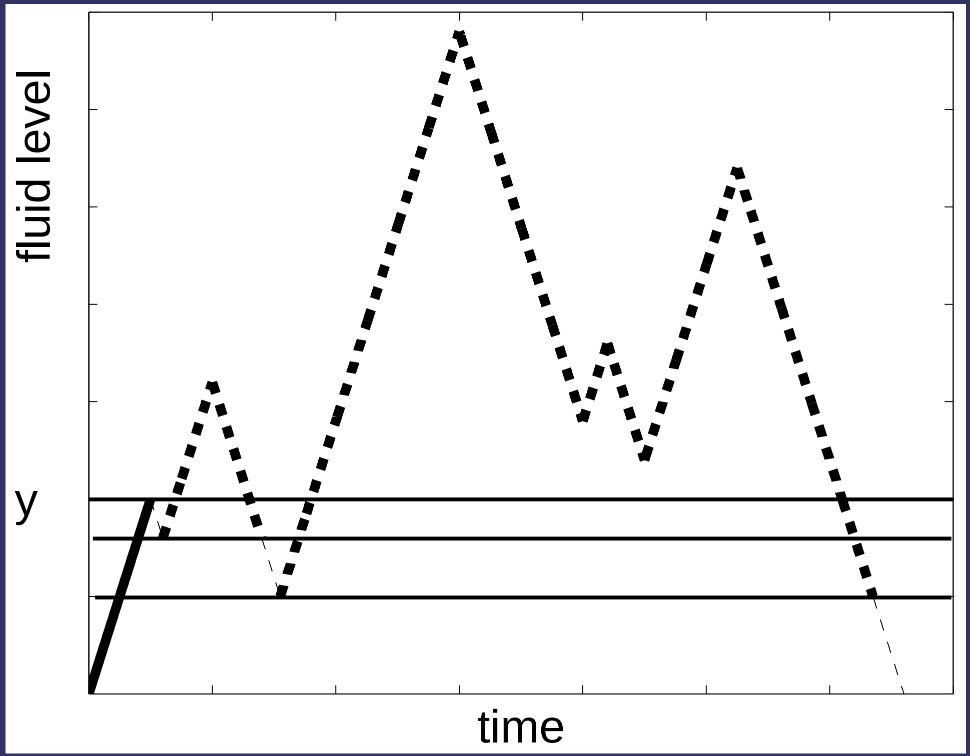
This expression is explicit. Recall that:

$$\mathcal{T} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, \quad (\widetilde{\nu_1}, \widetilde{\nu_2})[\mathcal{T} : \widetilde{e}] = [0 : 1]$$

and Ψ is the matrix recording the probabilities of return journey to the initial level.

Da Silva Soares and Latouche (2002)

Conditioning on the first epoch of decrease.



$$\Psi = \int_{y=0}^{\infty} e^{T_{11}y}\, T_{12}\, e^{(T_{22}+T_{21}\Psi)y} dy$$

Calculating Ψ

There are several equivalent integral-form formulae for Ψ.

Corollary:

Ψ is the minimal nonegative solution of the following Riccati equation

$$T_{12} + T_{11}\Psi + \Psi T_{21} + \Psi T_{12}\Psi = 0.$$

(For a general form of this result see Bean, O'Reilly and Taylor)

There are several different algorithms for Ψ.

Solving the Riccati equation for Ψ

Rewrite Riccati equation in an equivalent form:

$$(T_{11} + \Psi T_{21})\Psi + \Psi(T_{22} + T_{21}\Psi) = -T_{12} + \Psi T_{21}\Psi.$$

Algorithm (Newton's method, Guo 2001):

- $\Psi_0 = 0$,
- Ψ_{n+1} is the unique solution of the equation:

$$(T_{11} + \Psi_n T_{21})\Psi_{n+1} + \Psi_{n+1}(T_{22} + T_{21}\Psi_n) =$$
$$-T_{12} + \Psi_n T_{21}\Psi_n.$$

(Solving an equation of the form $AX + XB = D$ in each step)

Connection to QBDs

- Ramaswami (1999) maps a fluid model to a discrete-level QBD.
- Da Silva Soares and Latouche (2002) gives the physical intepretation of this construction.

Significance:

This construction allows for the calculation of the matrix Ψ using efficient algorithms for G in the QBDs.

QBD construction (Ramaswami 1999)

$$\text{Let } \vartheta \geq \max_{i \in \mathcal{S}} |\mathcal{T}_{ii}|, \quad P = I + \frac{1}{\vartheta}\mathcal{T} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}.$$

Consider QBD with transition matrices

$$A_0 = \begin{bmatrix} \frac{1}{2}I & 0 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} \frac{1}{2}P_{11} & 0 \\ P_{21} & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & \frac{1}{2}P_{12} \\ 0 & P_{22} \end{bmatrix}.$$

$$\text{Then } G = \begin{bmatrix} 0 & \Psi \\ 0 & P_{22} + P_{21}\Psi \end{bmatrix}.$$

Future directions

- Models with boundaries.
- Level-dependent models.
- Decision making component.
- Countable/continuous phase.
- Applications.
- . . .

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Submitted for publication.

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